NAME: ______________________________________ DATE: ________________________
Physics – Mr Smith *From Giancoli – Physics, 6th Edition*

PROBLEM SOLVING Adding Vectors

Here is a brief summary of how to add two or more vectors using components:

1. **Draw a diagram**, adding the vectors graphically by either the parallelogram or tail-to-tip method.
2. **Choose** x and y **axes**. Choose them in a way, if possible, that will make your work easier. (For example, choose one axis along the direction of one of the vectors so that vector will have only one component.)
3. **Resolve each vector** into its x and y components, showing each component along its appropriate (x or y) axis as a (dashed) arrow.
4. **Calculate each component** (when not given) using sines and cosines. If θ_1 is the angle that vector $\vec{\mathbf{V}}_1$ makes with the positive x axis, then:
$$V_{1x} = V_1 \cos\theta_1, \qquad V_{1y} = V_1 \sin\theta_1 .$$
Pay careful attention to **signs**: any component that points along the negative x or y axis gets a − sign.
5. **Add** the x **components** together to get the x component of the resultant. Ditto for y:
$$V_x = V_{1x} + V_{2x} + \text{any others}$$
$$V_y = V_{1y} + V_{2y} + \text{any others.}$$
This is the answer: the components of the resultant vector. Check signs to see if they fit the quadrant shown in your diagram (point 1 above).
6. If you want to know the **magnitude and direction** of the resultant vector, use Eqs. 3–4:
$$V = \sqrt{V_x^2 + V_y^2}, \qquad \tan\theta = \frac{V_y}{V_x}.$$
The vector diagram you already drew helps to obtain the correct position (quadrant) of the angle θ.

3–4 Adding Vectors by Components

Adding vectors graphically using a ruler and protractor is often not sufficiently accurate and is not useful for vectors in three dimensions. We discuss now a more powerful and precise method for adding vectors. But do not forget graphical methods—they are always useful for visualizing, for checking your math, and thus for getting the correct result.

Consider first a vector $\vec{\mathbf{V}}$ that lies in a particular plane. It can be expressed as the sum of two other vectors, called the **components** of the original vector. The components are usually chosen to be along two perpendicular directions. The process of finding the components is known as **resolving the vector into its components**. An example is shown in Fig. 3–10; the vector $\vec{\mathbf{V}}$ could be a displacement vector that points at an angle $\theta = 30°$ north of east, where we have chosen the positive x axis to be to the east and the positive y axis north. This vector $\vec{\mathbf{V}}$ is resolved into its x and y components by drawing dashed lines out from the tip (A) of the vector (lines AB and AC) making them perpendicular to the x and y axes. Then the lines 0B and 0C represent the x and y components of $\vec{\mathbf{V}}$, respectively, as shown in Fig. 3–10b. These **vector components** are written $\vec{\mathbf{V}}_x$ and $\vec{\mathbf{V}}_y$. We generally show vector components as arrows, like vectors, but dashed. The *scalar components*, V_x and V_y, are numbers, with units, that are given a positive or negative sign depending on whether they point along the positive or negative x or y axis. As can be seen in Fig. 3–10, $\vec{\mathbf{V}}_x + \vec{\mathbf{V}}_y = \vec{\mathbf{V}}$ by the parallelogram method of adding vectors.

Resolving a vector into components

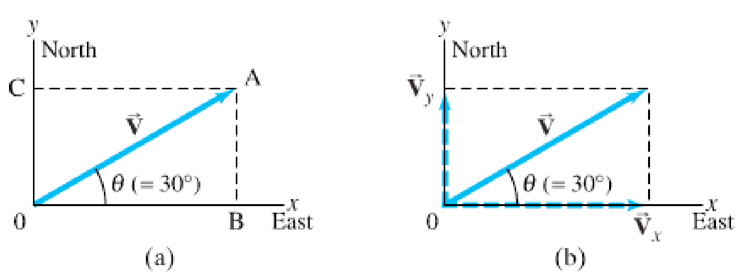
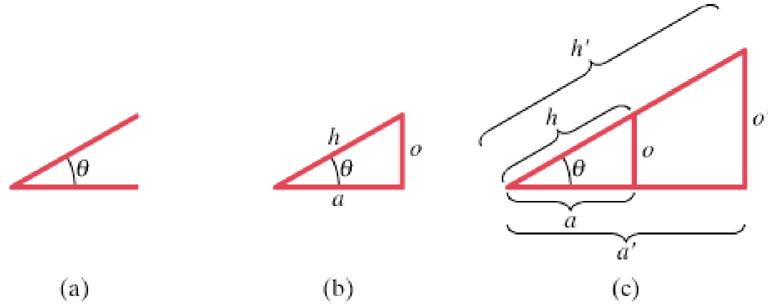


FIGURE 3–10 Resolving a vector $\vec{\mathbf{V}}$ into its components along an arbitrarily chosen set of x and y axes. The components, once found, themselves represent the vector. That is, the components contain as much information as the vector itself.

Space is made up of three dimensions, and sometimes it is necessary to resolve a vector into components along three mutually perpendicular directions. In rectangular coordinates the components are $\vec{\mathbf{V}}_x$, $\vec{\mathbf{V}}_y$, and $\vec{\mathbf{V}}_z$. Resolution of a vector in three dimensions is merely an extension of the above technique. We will mainly be concerned with situations in which the vectors are in a plane and two components are all that are necessary.

To add vectors using the method of components, we need to use the trigonometric functions sine, cosine, and tangent, which we now review.

FIGURE 3–11 Starting with an angle θ as in (a), we can construct right triangles of different sizes, (b) and (c), but the ratio of the lengths of the sides does not depend on the size of the triangle.



Given any angle θ, as in Fig. 3–11a, a right triangle can be constructed by drawing a line perpendicular to either of its sides, as in Fig. 3–11b. The longest side of a right triangle, opposite the right angle, is called the hypotenuse, which we label h. The side opposite the angle θ is labeled o, and the side adjacent is labeled a. We let h, o, and a represent the lengths of these sides, respectively. We now define the three trigonometric functions, sine, cosine, and tangent (abbreviated sin, cos, tan), in terms of the right triangle, as follows:

Trigonometric functions defined

$$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{o}{h}$$

$$\cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{a}{h} \qquad \textbf{(3–1)}$$

$$\tan\theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{o}{a}.$$

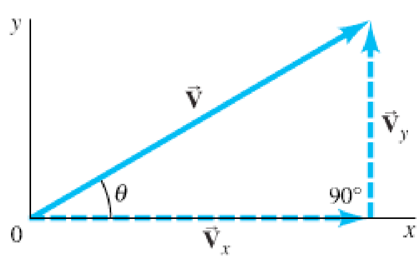
If we make the triangle bigger, but keep the same angles, then the ratio of the length of one side to the other, or of one side to the hypotenuse, remains the same. That is, in Fig. 3–11c we have: $a/h = a'/h'$; $o/h = o'/h'$; and $o/a = o'/a'$. Thus the values of sine, cosine, and tangent do not depend on how big the triangle is. They depend only on the size of the angle. The values of sine, cosine, and tangent for different angles can be found using a scientific calculator, or from the Table in Appendix A.

A useful trigonometric identity is

$$\sin^2\theta + \cos^2\theta = 1 \qquad \textbf{(3–2)}$$

which follows from the Pythagorean theorem ($o^2 + a^2 = h^2$ in Fig. 3–11). That is:

$$\sin^2\theta + \cos^2\theta = \frac{o^2}{h^2} + \frac{a^2}{h^2} = \frac{o^2 + a^2}{h^2} = \frac{h^2}{h^2} = 1.$$



$$\sin\theta = \frac{V_y}{V}$$

$$\cos\theta = \frac{V_x}{V}$$

$$\tan\theta = \frac{V_y}{V_x}$$

$$V^2 = V_x^2 + V_y^2$$

FIGURE 3–12 Finding the components of a vector using trigonometric functions.

(See also Appendix A for other details on trigonometric functions and identities.)

The use of trigonometric functions for finding the components of a vector is illustrated in Fig. 3–12, where a vector and its two components are thought of as making up a right triangle. We then see that the sine, cosine, and tangent are as given in the Figure. If we multiply the definition of $\sin\theta = V_y/V$ by V on both sides, we get

Components of a vector

$$V_y = V\sin\theta. \qquad \textbf{(3–3a)}$$

Similarly, from the definition of $\cos\theta$, we obtain

$$V_x = V\cos\theta. \qquad \textbf{(3–3b)}$$

Note that θ is chosen (by convention) to be the angle that the vector makes with the positive x axis.

Using Eqs. 3–3, we can calculate V_x and V_y for any vector, such as that illustrated in Fig. 3–10 or Fig. 3–12. Suppose $\vec{\mathbf{V}}$ represents a displacement of 500 m

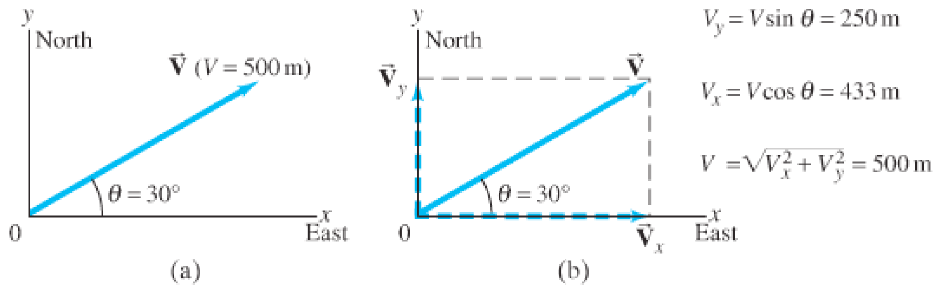


FIGURE 3–13 (a) Vector $\vec{\mathbf{V}}$ represents a displacement of 500 m at a 30° angle north of east. (b) The components of $\vec{\mathbf{V}}$ are $\vec{\mathbf{V}}_x$ and $\vec{\mathbf{V}}_y$, whose magnitudes are given on the right.

in a direction 30° north of east, as shown in Fig. 3–13. Then $V = 500\text{ m}$. From a calculator or Tables, $\sin 30° = 0.500$ and $\cos 30° = 0.866$. Then

$$V_x = V\cos\theta = (500\text{ m})(0.866) = 433\text{ m (east)},$$
$$V_y = V\sin\theta = (500\text{ m})(0.500) = 250\text{ m (north)}.$$

There are two ways to specify a vector in a given coordinate system:

Two ways to specify a vector

1. We can give its components, V_x and V_y.
2. We can give its magnitude V and the angle θ it makes with the positive x axis.

We can shift from one description to the other using Eqs. 3–3, and, for the reverse, by using the theorem of Pythagoras† and the definition of tangent:

Components related to magnitude and direction

$$V = \sqrt{V_x^2 + V_y^2} \tag{3–4a}$$

$$\tan\theta = \frac{V_y}{V_x} \tag{3–4b}$$

as can be seen in Fig. 3–12.

We can now discuss how to add vectors using components. The first step is to resolve each vector into its components. Next we can see, using Fig. 3–14, that the addition of any two vectors $\vec{\mathbf{V}}_1$ and $\vec{\mathbf{V}}_2$ to give a resultant, $\vec{\mathbf{V}} = \vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2$, implies that

Adding vectors analytically (by components)

$$\begin{aligned} V_x &= V_{1x} + V_{2x} \\ V_y &= V_{1y} + V_{2y}. \end{aligned} \tag{3–5}$$

That is, the sum of the x components equals the x component of the resultant, and similarly for y. That this is valid can be verified by a careful examination of Fig. 3–14. But note that we add all the x components together to get the x component of the resultant; and we add all the y components together to get the y component of the resultant. We do *not* add x components to y components.

If the magnitude and direction of the resultant vector are desired, they can be obtained using Eqs. 3–4.

† In three dimensions, the theorem of Pythagoras becomes $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$, where V_z is the component along the third, or z, axis.

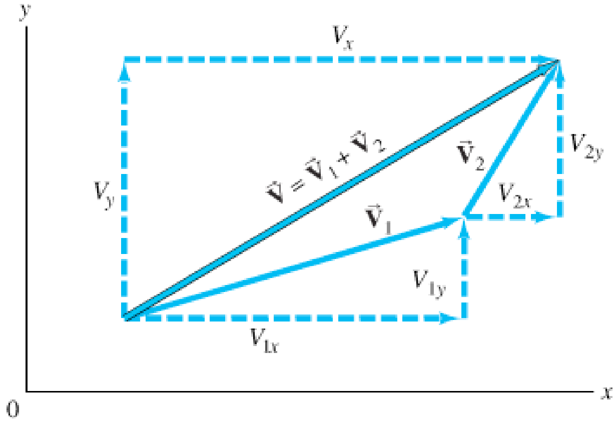
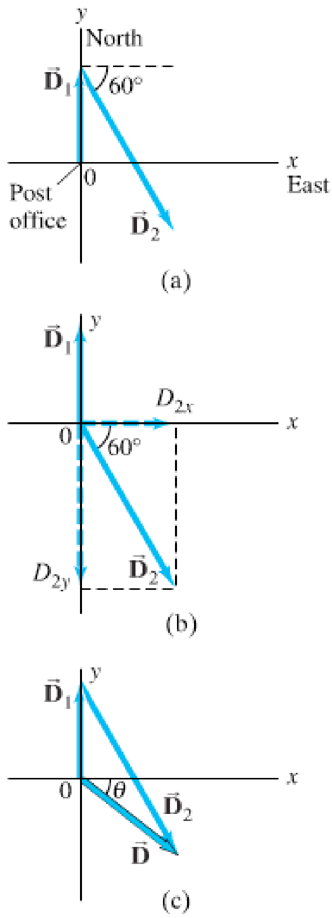


FIGURE 3–14 The components of $\vec{\mathbf{V}} = \vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2$ are $V_x = V_{1x} + V_{2x}$ and $V_y = V_{1y} + V_{2y}$.

Choice of axes can simplify effort needed

The components of a given vector will be different for different choices of coordinate axes. The choice of coordinate axes is always arbitrary. You can often reduce the work involved in adding vectors by a good choice of axes—for example, by choosing one of the axes to be in the same direction as one of the vectors. Then that vector will have only one nonzero component.



IGURE 3–15 Example 3–2. a) The two displacement vectors, $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$. (b) $\vec{\mathbf{D}}_2$ is resolved into :s components. (c) $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$ are dded graphically to obtain he resultant $\vec{\mathbf{D}}$. The component nethod of adding the vectors is xplained in the Example.

EXAMPLE 3–2 **Mail carrier's displacement.** A rural mail carrier leaves the post office and drives 22.0 km in a northerly direction. She then drives in a direction 60.0° south of east for 47.0 km (Fig. 3–15a). What is her displacement from the post office?

APPROACH We resolve each vector into its x and y components. We add the x components together, and then the y components together, giving us the x and y components of the resultant. We choose the positive x axis to be east and the positive y axis to be north, since those are the compass directions used on most maps.

SOLUTION Resolve each displacement vector into its components, as shown in Fig. 3–15b. Since $\vec{\mathbf{D}}_1$ has magnitude 22.0 km and points north, it has only a y component:

$$D_{1x} = 0, \qquad D_{1y} = 22.0\,\text{km}.$$

$\vec{\mathbf{D}}_2$ has both x and y components:

$$\begin{aligned} D_{2x} &= +(47.0\,\text{km})(\cos 60^\circ) = +(47.0\,\text{km})(0.500) = +23.5\,\text{km} \\ D_{2y} &= -(47.0\,\text{km})(\sin 60^\circ) = -(47.0\,\text{km})(0.866) = -40.7\,\text{km}. \end{aligned}$$

Notice that D_{2y} is negative because this vector component points along the negative y axis. The resultant vector, $\vec{\mathbf{D}}$, has components:

$$\begin{aligned} D_x &= D_{1x} + D_{2x} = 0\,\text{km} + 23.5\,\text{km} = +23.5\,\text{km} \\ D_y &= D_{1y} + D_{2y} = 22.0\,\text{km} + (-40.7\,\text{km}) = -18.7\,\text{km}. \end{aligned}$$

This specifies the resultant vector completely:

$$D_x = 23.5\,\text{km}, \qquad D_y = -18.7\,\text{km}.$$

We can also specify the resultant vector by giving its magnitude and angle using Eqs. 3–4:

$$D = \sqrt{D_x^2 + D_y^2} = \sqrt{(23.5\,\text{km})^2 + (-18.7\,\text{km})^2} = 30.0\,\text{km}$$

$$\tan\theta = \frac{D_y}{D_x} = \frac{-18.7\,\text{km}}{23.5\,\text{km}} = -0.796.$$

A calculator with an INV TAN, an ARC TAN, or a TAN^{-1} key gives $\theta = \tan^{-1}(-0.796) = -38.5^\circ$. The negative sign means $\theta = 38.5^\circ$ below the x axis, Fig. 3–15c. So, the resultant displacement is 30.0 km directed at 38.5° in a southeasterly direction.

NOTE Always be attentive about the quadrant in which the resultant vector lies. An electronic calculator does not fully give this information, but a good diagram does.

PROBLEM SOLVING

Identify the correct quadrant by drawing a careful diagram

The signs of trigonometric functions depend on which "quadrant" the angle falls in: for example, the tangent is positive in the first and third quadrants (from 0° to 90°, and 180° to 270°), but negative in the second and fourth quadrants; see Appendix A–7. The best way to keep track of angles, and to check any vector result, is always to draw a vector diagram. A vector diagram gives you something tangible to look at when analyzing a problem, and provides a check on the results.

EXAMPLE 3–3 **Three short trips.** An airplane trip involves three legs, with two stopovers, as shown in Fig. 3–16a. The first leg is due east for 620 km; the second leg is southeast (45°) for 440 km; and the third leg is at 53° south of west, for 550 km, as shown. What is the plane's total displacement?

APPROACH We follow the steps in the above Problem Solving Box.

SOLUTION

1. **Draw a diagram** such as Fig. 3–16a, where $\vec{\mathbf{D}}_1$, $\vec{\mathbf{D}}_2$, and $\vec{\mathbf{D}}_3$ represent the three legs of the trip, and $\vec{\mathbf{D}}_R$ is the plane's total displacement.
2. **Choose axes**: Axes are also shown in Fig. 3–16a.
3. **Resolve components**: It is imperative to draw a good figure. The components are drawn in Fig. 3–16b. Instead of drawing all the vectors starting from a common origin, as we did in Fig. 3–15b, here we draw them "tail-to-tip" style, which is just as valid and may make it easier to see.
4. **Calculate the components**:

$$\begin{aligned}
\vec{\mathbf{D}}_1: D_{1x} &= +D_1 \cos 0^\circ &&= D_1 = 620\text{ km} \\
D_{1y} &= +D_1 \sin 0^\circ &&= 0\text{ km} \\
\vec{\mathbf{D}}_2: D_{2x} &= +D_2 \cos 45^\circ &&= +(440\text{ km})(0.707) = +311\text{ km} \\
D_{2y} &= -D_2 \sin 45^\circ &&= -(440\text{ km})(0.707) = -311\text{ km} \\
\vec{\mathbf{D}}_3: D_{3x} &= -D_3 \cos 53^\circ &&= -(550\text{ km})(0.602) = -331\text{ km} \\
D_{3y} &= -D_3 \sin 53^\circ &&= -(550\text{ km})(0.799) = -439\text{ km.}
\end{aligned}$$

   We have given a minus sign to each component that in Fig. 3–16b points in the $-x$ or $-y$ direction. The components are shown in the Table in the margin.
5. **Add the components**: We add the x components together, and we add the y components together to obtain the x and y components of the resultant:

$$\begin{aligned}
D_x &= D_{1x} + D_{2x} + D_{3x} = 620\text{ km} + 311\text{ km} - 331\text{ km} = 600\text{ km} \\
D_y &= D_{1y} + D_{2y} + D_{3y} = 0\text{ km} - 311\text{ km} - 439\text{ km} = -750\text{ km.}
\end{aligned}$$

   The x and y components are 600 km and -750 km, and point respectively to the east and south. This is one way to give the answer.
6. **Magnitude and direction**: We can also give the answer as

$$D_R = \sqrt{D_x^2 + D_y^2} = \sqrt{(600)^2 + (-750)^2}\text{ km} = 960\text{ km}$$

$$\tan\theta = \frac{D_y}{D_x} = \frac{-750\text{ km}}{600\text{ km}} = -1.25, \qquad \text{so } \theta = -51^\circ.$$

   Thus, the total displacement has magnitude 960 km and points 51° below the x axis (south of east), as was shown in our original sketch, Fig. 3–16a.

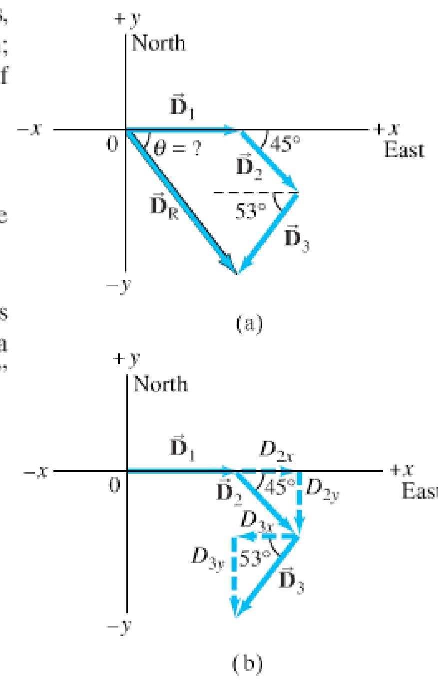


FIGURE 3–16 Example 3–3.

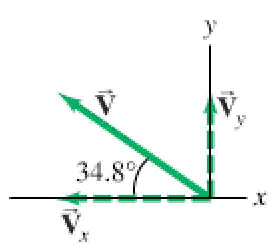
Vector	Components x (km)	Components y (km)
$\vec{\mathbf{D}}_1$	620	0
$\vec{\mathbf{D}}_2$	311	−311
$\vec{\mathbf{D}}_3$	−331	−439
$\vec{\mathbf{D}}_R$	600	−750

Now try some VECTOR PROBLEMS – Draw a nice big diagram – don't be afraid to use up space!

7. (II) $\vec{\mathbf{V}}$ is a vector 14.3 units in magnitude and points at an angle of 34.8° above the negative x axis. (a) Sketch this vector. (b) Find V_x and V_y. (c) Use V_x and V_y to obtain (again) the magnitude and direction of $\vec{\mathbf{V}}$. [*Note*: Part (c) is a good way to check if you've resolved your vector correctly.]

8. (II) Vector $\vec{\mathbf{V}}_1$ is 6.6 units long and points along the negative x axis. Vector $\vec{\mathbf{V}}_2$ is 8.5 units long and points at +45° to the positive x axis. (a) What are the x and y components of each vector? (b) Determine the sum $\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2$ (magnitude and angle).

7. (a)



(b) −11.7 units, 8.16 units;

(c) 14.3 units, 34.8° above the $-x$ axis.

9. (II) An airplane is traveling 735 km/h in a direction 41.5° west of north (Fig. 3–31). (*a*) Find the components of the velocity vector in the northerly and westerly directions. (*b*) How far north and how far west has the plane traveled after 3.00 h?

9. (*a*) 550 km/h, 487 km/h; (*b*) 1650 km, 1460 km.

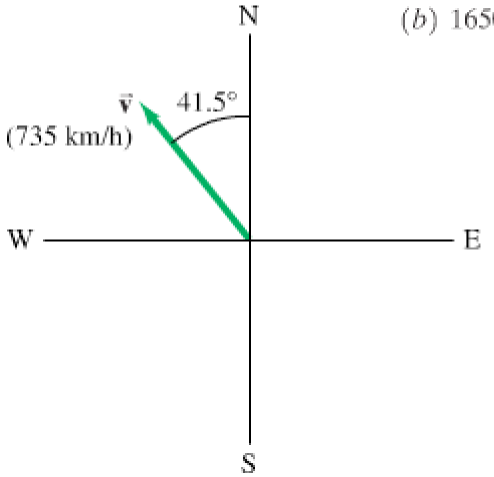


FIGURE 3–31 Problem 9.

10. (II) Three vectors are shown in Fig. 3–32. Their magnitudes are given in arbitrary units. Determine the sum of the three vectors. Give the resultant in terms of (*a*) components, (*b*) magnitude and angle with the *x* axis.

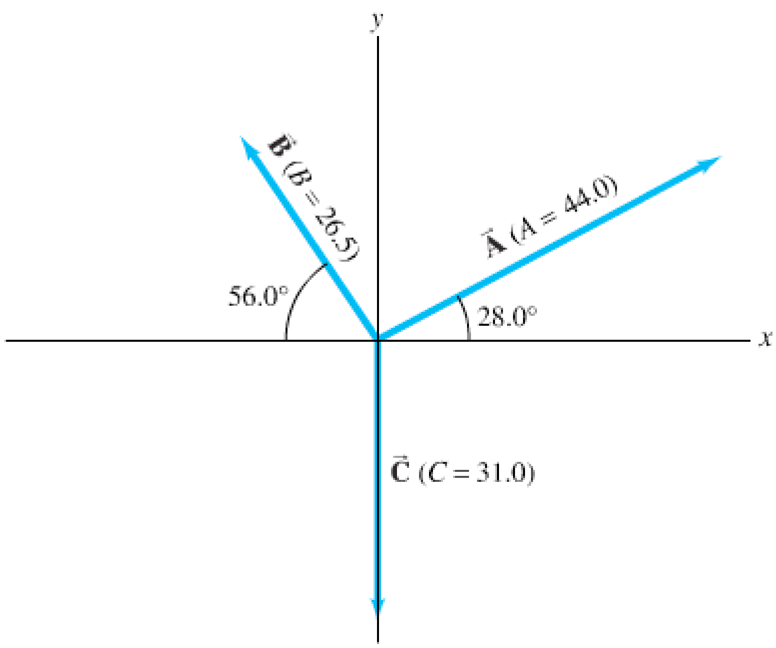


FIGURE 3–32 Problems 10, 11, 12, 13, and 14. Vector magnitudes are given in arbitrary units.

11. (II) Determine the vector $\vec{A} - \vec{C}$, given the vectors $\vec{A}$ and $\vec{C}$ in Fig. 3–32.

12. (II) (*a*) Given the vectors $\vec{A}$ and $\vec{B}$ shown in Fig. 3–32, determine $\vec{B} - \vec{A}$. (*b*) Determine $\vec{A} - \vec{B}$ without using your answer in (*a*). Then compare your results and see if they are opposite.

13. (II) For the vectors given in Fig. 3–32, determine (*a*) $\vec{A} - \vec{B} + \vec{C}$, (*b*) $\vec{A} + \vec{B} - \vec{C}$, and (*c*) $\vec{C} - \vec{A} - \vec{B}$.

14. (II) For the vectors shown in Fig. 3–32, determine (*a*) $\vec{B} - 2\vec{A}$, (*b*) $2\vec{A} - 3\vec{B} + 2\vec{C}$.

11. 64.6, 53.1°.

13. (*a*) 62.6, 329°; (*b*) 77.5, 71.9°; (*c*) 77.5, 251.9°.