Measurement and uncertainties 1

1.1 Measurement in physics

Learning objectives

- State the fundamental units of the SI system.
- Be able to express numbers in scientific notation.
- Appreciate the order of magnitude of various quantities.
- Perform simple order-of-magnitude calculations mentally.
- Express results of calculations to the correct number of significant figures.

Physics is an experimental science in which measurements made must be expressed in units. In the international system of units used throughout this book, the SI system, there are seven fundamental units, which are defined in this section. All quantities are expressed in terms of these units directly, or as a combination of them.

The SI system

The SI system (short for Système International d'Unités) has seven **fundamental units** (it is quite amazing that only seven are required). These are:

1 The **metre** (m). This is the unit of distance. It is the distance travelled by light in a vacuum in a time of $\frac{1}{299\,792\,458}$ seconds.

2 The **kilogram** (kg). This is the unit of mass. It is the mass of a certain quantity of a platinum–iridium alloy kept at the Bureau International des Poids et Mesures in France.

3 The **second** (s). This is the unit of time. A second is the duration of 9 192 631 770 full oscillations of the electromagnetic radiation emitted in a transition between the two hyperfine energy levels in the ground state of a caesium-133 atom.

4 The **ampere** (A). This is the unit of electric current. It is defined as that current which, when flowing in two parallel conductors 1 m apart, produces a force of 2×10^{7} N on a length of 1 m of the conductors.

5 The **kelvin** (K). This is the unit of temperature. It is $\frac{1}{273.16}$ of the thermodynamic temperature of the triple point of water.

6 The **mole** (mol). One mole of a substance contains as many particles as there are atoms in 12 g of carbon-12. This special number of particles is called Avogadro's number and is approximately 6.02×10^{23}.

7 The **candela** (cd). This is a unit of luminous intensity. It is the intensity of a source of frequency 5.40×10^{14} Hz emitting $\frac{1}{683}$ W per steradian.

You do not need to memorise the details of these definitions.

In this book we will use all of the basic units except the last one. Physical quantities other than those above have units that are combinations of the seven fundamental units. They have **derived** units. For example, speed has units of distance over time, metres per second (i.e. m/s or, preferably, m s^{-1}). Acceleration has units of metres per second squared (i.e. m/s^2, which we write as m s^{-2}). Similarly, the unit of force is the newton (N). It equals the combination kg m s^{-2}. Energy, a very important quantity in physics, has the joule (J) as its unit. The joule is the combination N m and so equals ($\text{kg m s}^{-2}\,\text{m}$), or $\text{kg m}^2\,\text{s}^{-2}$. The quantity power has units of energy per unit of time, and so is measured in $J\,s^{-1}$. This combination is called a watt. Thus:

$$1\,W = (1\,N\,m\,s^{-1}) = (1\,kg\,m\,s^{-2}\,m\,s^{-1}) = 1\,kg\,m^2\,s^{-3}$$

Metric multipliers

Small or large quantities can be expressed in terms of units that are related to the basic ones by powers of 10. Thus, a nanometre (nm) is 10^{-9} m, a microgram (μg) is $10^{-6}\,g = 10^{-9}\,kg$, a gigaelectron volt (GeV) equals 10^9 eV, etc. The most common prefixes are given in Table **1.1**.

Power	Prefix	Symbol	Power	Prefix	Symbol
10^{-18}	atto-	A	10^{1}	deka-	da
10^{-15}	femto-	F	10^{2}	hecto-	h
10^{-12}	pico-	p	10^{3}	kilo-	k
10^{-9}	nano-	n	10^{6}	mega-	M
10^{-6}	micro-	μ	10^{9}	giga-	G
10^{-3}	milli-	m	10^{12}	tera-	T
10^{-2}	centi-	c	10^{15}	peta-	P
10^{-1}	deci-	d	10^{18}	exa-	E

Table 1.1 Common prefixes in the SI system.

Orders of magnitude and estimates

Expressing a quantity as a plain power of 10 gives what is called the **order of magnitude** of that quantity. Thus, the mass of the universe has an order of magnitude of 10^{53} kg and the mass of the Milky Way galaxy has an order of magnitude of 10^{41} kg. The ratio of the two masses is then simply 10^{12}.

Tables **1.2**, **1.3** and **1.4** give examples of distances, masses and times, given as orders of magnitude.

	Length / m
distance to edge of observable universe	10^{26}
distance to the Andromeda galaxy	10^{22}
diameter of the Milky Way galaxy	10^{21}
distance to nearest star	10^{16}
diameter of the solar system	10^{13}
distance to the Sun	10^{11}
radius of the Earth	10^{7}
size of a cell	10^{-5}
size of a hydrogen atom	10^{-10}
size of an $A = 50$ nucleus	10^{-15}
size of a proton	10^{-15}
Planck length	10^{-35}

Table 1.2 Some interesting distances.

	Mass / kg
the universe	10^{53}
the Milky Way galaxy	10^{41}
the Sun	10^{30}
the Earth	10^{24}
Boeing 747 (empty)	10^{5}
an apple	0.2
a raindrop	10^{-6}
a bacterium	10^{-15}
smallest virus	10^{-21}
a hydrogen atom	10^{-27}
an electron	10^{-30}

Table 1.3 Some interesting masses.

	Time / s
age of the universe	10^{17}
age of the Earth	10^{17}
time of travel by light to nearby star	10^{8}
one year	10^{7}
one day	10^{5}
period of a heartbeat	1
lifetime of a pion	10^{-8}
lifetime of the omega particle	10^{-10}
time of passage of light across a proton	10^{-24}

Table 1.4 Some interesting times.

Worked examples

1.1 Estimate how many grains of sand are required to fill the volume of the Earth. (This is a classic problem that goes back to Aristotle. The radius of the Earth is about 6×10^6 m.)

The volume of the Earth is:

$$\frac{4}{3}\pi R^3 \approx \frac{4}{3} \times 3 \times (6 \times 10^6)^3 \approx 8 \times 10^{20} \approx 10^{21}\,\mathrm{m}^3$$

The diameter of a grain of sand varies of course, but we will take 1 mm as a fair estimate. The volume of a grain of sand is about $(1 \times 10^{-3})^3\,\mathrm{m}^3$.

Then the number of grains of sand required to fill the Earth is:

$$\frac{10^{21}}{(1 \times 10^{-3})^3} \approx 10^{30}$$

1.2 Estimate the speed with which human hair grows.

I have my hair cut every two months and the barber cuts a length of about 2 cm. The speed is therefore:

$$\frac{2 \times 10^{-2}}{2 \times 30 \times 24 \times 60 \times 60}\,\mathrm{m\,s^{-1}} \approx \frac{10^{-2}}{3 \times 2 \times 36 \times 10^4}$$

$$\approx \frac{10^{-6}}{6 \times 40} = \frac{10^{-6}}{240}$$

$$\approx 4 \times 10^{-9}\,\mathrm{m\,s^{-1}}$$

1.3 Estimate how long the line would be if all the people on Earth were to hold hands in a straight line. Calculate how many times it would wrap around the Earth at the equator. (The radius of the Earth is about 6×10^6 m.)

Assume that each person has his or her hands stretched out to a distance of 1.5 m and that the population of Earth is 7×10^9 people.

Then the length of the line of people would be $7 \times 10^9 \times 1.5\,\text{m} = 10^{10}\,\text{m}$.

The circumference of the Earth is $2\pi R \approx 6 \times 6 \times 10^6\,\text{m} \approx 4 \times 10^7\,\text{m}$.

So the line would wrap $\frac{10^{10}}{4 \times 10^7} \approx 250$ times around the equator.

1.4 Estimate how many apples it takes to have a combined mass equal to that of an ordinary family car.

Assume that an apple has a mass of 0.2 kg and a car has a mass of 1400 kg.

Then the number of apples is $\frac{1400}{0.2} = 7 \times 10^3$.

1.5 Estimate the time it takes light to arrive at Earth from the Sun. (The Earth–Sun distance is 1.5×10^{11} m.)

The time taken is $\frac{\text{distance}}{\text{speed}} = \frac{1.5 \times 10^{11}}{3 \times 10^8} \approx 0.5 \times 10^4 = 500\,\text{s} \approx 8\,\text{min}$

Significant figures

The **number** of digits used to express a number carries information about how precisely the number is known. A stopwatch reading of 3.2 s (two significant figures, s.f.) is less precise than a reading of 3.23 s (three s.f.). If you are told what your salary is going to be, you would like that number to be known as precisely as possible. It is less satisfying to be told that your salary will be 'about 1000' (1 s.f.) euro a month compared to a salary of 'about 1250' (3 s.f.) euro a month. Not because 1250 is larger than 1000 but because the number of 'about 1000' could mean anything from a low of 500 to a high of 1500. You could be lucky and get the 1500 but you cannot be sure. With a salary of 'about 1250' your actual salary could be anything from 1200 to 1300, so you have a pretty good idea of what it will be.

How to find the number of significant figures in a number is illustrated in Table **1.5**.

Number	Number of s.f.	Reason	Scientific notation
504	3	in an integer all digits count (if last digit is not zero)	5.04×10^2
608 000	3	zeros at the end of an integer do not count	6.08×10^5
200	1	zeros at the end of an integer do not count	2×10^2
0.000 305	3	zeros in front do not count	3.05×10^{-4}
0.005 900	4	zeros at the end of a decimal count, those in front do not	5.900×10^{-3}

Table 1.5 Rules for significant figures.

Scientific notation means writing a number in the form $a\times10^b$, where a is decimal such that $1\le a<10$ and b is a positive or negative integer. The number of digits in a is the number of significant figures in the number.

In multiplication or division (or in raising a number to a power or taking a root), the result must have as many significant figures as the **least** precisely known number entering the calculation. So we have that:

$$\underbrace{23}_{2\text{ s.f.}}\times\underbrace{578}_{3\text{ s.f.}}=13\,294\approx\underbrace{1.3\times10^4}_{2\text{ s.f.}}$$ (the least number of s.f. is shown in red)

$$\frac{\underbrace{6.244}_{4\text{ s.f.}}}{\underbrace{1.25}_{3\text{ s.f.}}}=4.9952\ldots\approx\underbrace{5.00\times10^0}_{3\text{ s.f.}}=5.00$$

$$\underbrace{12.3^3}_{3\text{ s.f.}}=1860.867\ldots\approx\underbrace{1.86\times10^3}_{3\text{ s.f.}}$$

$$\underbrace{\sqrt{58900}}_{3\text{ s.f.}}=242.6932\ldots\approx\underbrace{2.43\times10^2}_{3\text{ s.f.}}$$

In adding and subtracting, the number of decimal digits in the answer must be equal to the least number of decimal places in the numbers added or subtracted. Thus:

$$\underbrace{3.21}_{2\text{ d.p.}}+\underbrace{4.1}_{1\text{ d.p.}}=7.32\approx\underbrace{7.3}_{1\text{ d.p.}}$$ (the least number of d.p. is shown in red)

$$\underbrace{12.367}_{3\text{ d.p.}}-\underbrace{3.15}_{2\text{ d.p.}}=9.217\approx\underbrace{9.22}_{2\text{ d.p.}}$$

Use the rules for rounding when writing values to the correct number of decimal places or significant figures. For example, the number $542.48=5.4248\times10^2$ rounded to 2, 3 and 4 s.f. becomes:

$5.4|248\times10^2\approx5.4\times10^2$ rounded to 2 s.f.
$5.42|48\times10^2\approx5.42\times10^2$ rounded to 3 s.f.
$5.424|8\times10^2\approx5.425\times10^2$ rounded to 4 s.f.

There is a special rule for rounding when the last digit to be dropped is 5 and it is followed only by zeros, or not followed by any other digit.

This is the odd–even rounding rule. For example, consider the number 3.250 000 0... where the zeros continue indefinitely. How does this number round to 2 s.f.? Because the digit before the 5 is **even** we do not round up, so 3.250 000 0... becomes 3.2. But 3.350 000 0... rounds up to 3.4 because the digit before the 5 is **odd**.

Nature of science

Early work on electricity and magnetism was hampered by the use of different systems of units in different parts of the world. Scientists realised they needed to have a common system of units in order to learn from each other's work and reproduce experimental results described by others. Following an international review of units that began in 1948, the SI system was introduced in 1960. At that time there were six base units. In 1971 the mole was added, bringing the number of base units to the seven in use today.

As the instruments used to measure quantities have developed, the definitions of standard units have been refined to reflect the greater precision possible. Using the transition of the caesium-133 atom to measure time has meant that smaller intervals of time can be measured accurately. The SI system continues to evolve to meet the demands of scientists across the world. Increasing precision in measurement allows scientists to notice smaller differences between results, but there is always uncertainty in any experimental result. There are no 'exact' answers.

? Test yourself

1 How long does light take to travel across a proton?

2 How many hydrogen atoms does it take to make up the mass of the Earth?

3 What is the age of the universe expressed in units of the Planck time?

4 How many heartbeats are there in the lifetime of a person (75 years)?

5 What is the mass of our galaxy in terms of a solar mass?

6 What is the diameter of our galaxy in terms of the astronomical unit, i.e. the distance between the Earth and the Sun ($1\,\text{AU} = 1.5 \times 10^{11}\,\text{m}$)?

7 The molar mass of water is $18\,\text{g}\,\text{mol}^{-1}$. How many molecules of water are there in a glass of water (mass of water 300 g)?

8 Assuming that the mass of a person is made up entirely of water, how many molecules are there in a human body (of mass 60 kg)?

9 Give an order-of-magnitude estimate of the density of a proton.

10 How long does light take to traverse the diameter of the solar system?

11 An electron volt (eV) is a unit of energy equal to $1.6 \times 10^{-19}\,\text{J}$. An electron has a kinetic energy of 2.5 eV.

a How many joules is that?

b What is the energy in eV of an electron that has an energy of $8.6 \times 10^{-18}\,\text{J}$?

12 What is the volume in cubic metres of a cube of side 2.8 cm?

13 What is the side in metres of a cube that has a volume of 588 cubic millimetres?

14 Give an order-of-magnitude estimate for the mass of:

a an apple

b this physics book

c a soccer ball.

15 A white dwarf star has a mass about that of the Sun and a radius about that of the Earth. Give an order-of-magnitude estimate of the density of a white dwarf.

16 A sports car accelerates from rest to 100 km per hour in 4.0 s. What fraction of the acceleration due to gravity is the car's acceleration?

17 Give an order-of-magnitude estimate for the number of electrons in your body.

18 Give an order-of-magnitude estimate for the ratio of the electric force between two electrons 1 m apart to the gravitational force between the electrons.

19 The frequency f of oscillation (a quantity with units of inverse seconds) of a mass m attached to a spring of spring constant k (a quantity with units of force per length) is related to m and k. By writing $f = cm^x k^y$ and matching units on both sides, show that $f = c\sqrt{\frac{k}{m}}$, where c is a dimensionless constant.

20 A block of mass 1.2 kg is raised a vertical distance of 5.55 m in 2.450 s. Calculate the power delivered. ($P = \frac{mgh}{t}$ and $g = 9.81\,\text{m s}^{-2}$)

21 Find the kinetic energy ($E_K = \frac{1}{2}mv^2$) of a block of mass 5.00 kg moving at a speed of $12.5\,\text{m s}^{-1}$.

22 Without using a calculator, **estimate** the value of the following expressions. Then compare your estimate with the exact value found using a calculator.

a $\frac{243}{43}$

b 2.80×1.90

c $312 \times \frac{480}{160}$

d $\frac{8.99 \times 10^9 \times 7 \times 10^{-16} \times 7 \times 10^{-6}}{(8 \times 10^2)^2}$

e $\frac{6.6 \times 10^{-11} \times 6 \times 10^{24}}{(6.4 \times 10^6)^2}$

1.2 Uncertainties and errors

This section introduces the basic methods of dealing with experimental error and uncertainty in measured physical quantities. Physics is an experimental science and often the experimenter will perform an experiment to test the prediction of a given theory. No measurement will ever be completely accurate, however, and so the result of the experiment will be presented with an experimental error.

Learning objectives

- Distinguish between random and systematic uncertainties.
- Work with absolute, fractional and percentage uncertainties.
- Use error bars in graphs.
- Calculate the uncertainty in a gradient or an intercept.

Types of uncertainty

There are two main types of uncertainty or error in a measurement. They can be grouped into **systematic** and **random**, although in many cases it is not possible to distinguish clearly between the two. We may say that random uncertainties are almost always the fault of the observer, whereas systematic errors are due to both the observer and the instrument being used. In practice, all uncertainties are a combination of the two.

Systematic errors

A **systematic error** biases measurements in the same direction; the measurements are always too large or too small. If you use a metal ruler to measure length on a very hot day, all your length measurements will be too small because the metre ruler expanded in the hot weather. If you use an ammeter that shows a current of 0.1 A even before it is connected to a circuit, every measurement of current made with this ammeter will be larger than the true value of the current by 0.1 A.

Suppose you are investigating Newton's second law by measuring the acceleration of a cart as it is being pulled by a falling weight of mass m (Figure **1.1**). Almost certainly there is a frictional force f between the cart and the table surface. If you forget to take this force into account, you would expect the cart's acceleration a to be:

$$a = \frac{mg}{M}$$

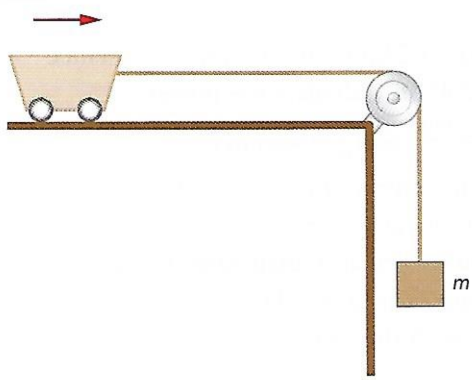


Figure 1.1 The falling block accelerates the cart.

where M is the constant combined mass of the cart and the falling block.

The graph of the acceleration versus m would be a straight line through the origin, as shown by the red line in Figure **1.2**. If you actually do the experiment, you will find that you do get a straight line, but not through the origin (blue line in Figure **1.2**). There is a negative intercept on the vertical axis.

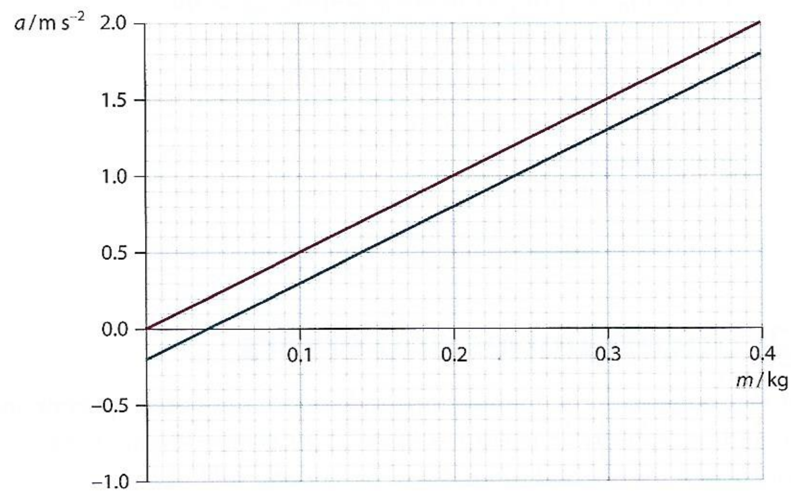


Figure 1.2 The variation of acceleration with falling mass with (blue) and without (red) frictional forces.

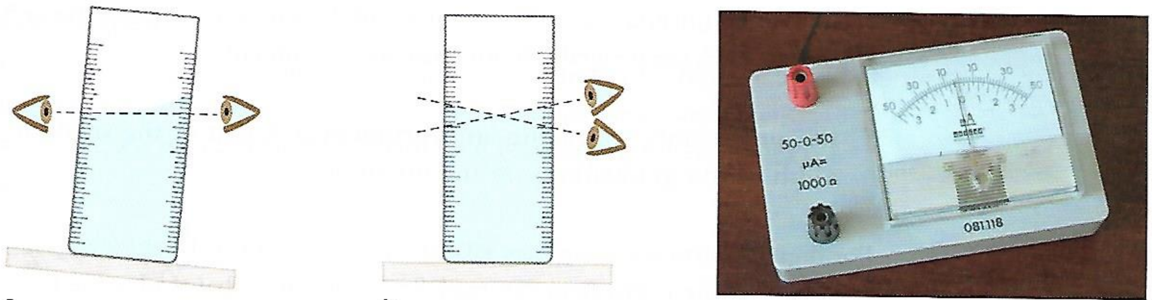
This is because with the frictional force present, Newton's second law predicts that:

$$a = \frac{mg}{M} - \frac{f}{M}$$

So a graph of acceleration a versus mass m would give a straight line with a negative intercept on the vertical axis.

Systematic errors can result from the technique used to make a measurement. There will be a systematic error in measuring the volume of a liquid inside a graduated cylinder if the tube is not exactly vertical. The measured values will always be larger or smaller than the true value, depending on which side of the cylinder you look at (Figure **1.3a**). There will also be a systematic error if your eyes are not aligned with the liquid level in the cylinder (Figure **1.3b**). Similarly, a systematic error will arise if you do not look at an analogue meter directly from above (Figure **1.3c**).

Systematic errors are hard to detect and take into account.



a b c

Figure 1.3 Parallax errors in measurements.

Random uncertainties

The presence of **random uncertainty** is revealed when repeated measurements of the same quantity show a spread of values, some too large some too small. Unlike systematic errors, which are always biased to be in the same direction, random uncertainties are unbiased. Suppose you ask ten people to use stopwatches to measure the time it takes an athlete to run a distance of 100 m. They stand by the finish line and start their stopwatches when the starting pistol fires. You will most likely get ten different values for the time. This is because some people will start/stop the stopwatches too early and some too late. You would expect that if you took an average of the ten times you would get a better **estimate** for the time than any of the individual measurements: the measurements **fluctuate** about some value. Averaging a large number of measurements gives a more **accurate** estimate of the result. (See the section on accuracy and precision, overleaf.)

We include within random uncertainties, **reading** uncertainties (which really is a different type of error altogether). These have to do with the precision with which we can read an instrument. Suppose we use a ruler to record the position of the right end of an object, Figure **1.4**.

Figure 1.4 Two rulers with different graduations. The top has a width between graduations of 0.2 cm and the other 0.1 cm.

The first ruler has graduations separated by 0.2 cm. We are confident that the position of the right end is greater than 23.2 cm and smaller than 23.4 cm. The true value is somewhere between these bounds. The average of the lower and upper bounds is 23.3 cm and so we quote the measurement as (23.3 ± 0.1) cm. Notice that the uncertainty of ± 0.1 cm **is half the smallest width** on the ruler. This is the conservative way of doing things and not everyone agrees with this. What if you scanned the diagram in Figure **1.4** on your computer, enlarged it and used your computer to draw further lines in between the graduations of the ruler. Then you could certainly read the position to better precision than the ± 0.1 cm. Others might claim that they can do this even without a computer or a scanner! They might say that the right end is definitely short of the 23.3 cm point. We will not discuss this any further – it is an endless discussion and, at this level, pointless.

Now let us use a ruler with a finer scale. We are again confident that the position of the right end is greater than 32.3 cm and smaller than 32.4 cm. The true value is somewhere between these bounds. The average of the bounds is 32.35 cm so we quote a measurement of (32.35 ± 0.05) cm. Notice again that the uncertainty of ± 0.05 cm is half the smallest width on the ruler. This gives the general rule for analogue instruments:

The uncertainty in reading an instrument is ± half of the smallest width of the graduations on the instrument.

Instrument	Reading error
ruler	±0.5 mm
vernier calipers	±0.05 mm
micrometer	±0.005 mm
electronic weighing scale	±0.1 g
stopwatch	±0.01 s

Table 1.6 Reading errors for some common instruments.

For digital instruments, we may take the reading error to be the smallest division that the instrument can read. So a stopwatch that reads time to two decimal places, e.g. 25.38 s, will have a reading error of ± 0.01 s, and a weighing scale that records a mass as 184.5 g will have a reading error of ± 0.1 g. Typical reading errors for some common instruments are listed in Table **1.6**.

Accuracy and precision

In physics, a measurement is said to be **accurate** if the systematic error in the measurement is small. This means in practice that the measured value is very close to the accepted value for that quantity (assuming that this is known – it is not always). A measurement is said to be **precise** if the random uncertainty is small. This means in practice that when the measurement was repeated many times, the individual values were close to each other. We normally illustrate the concepts of accuracy and precision with the diagrams in Figure **1.5**: the red stars indicate individual measurements. The 'true' value is represented by the common centre of the three circles, the 'bull's-eye'. Measurements are precise if they are clustered together. They are accurate if they are close to the centre. The descriptions of three of the diagrams are obvious; the bottom right clearly shows results that are not precise because they are not clustered together. But they are accurate because their average value is roughly in the centre.

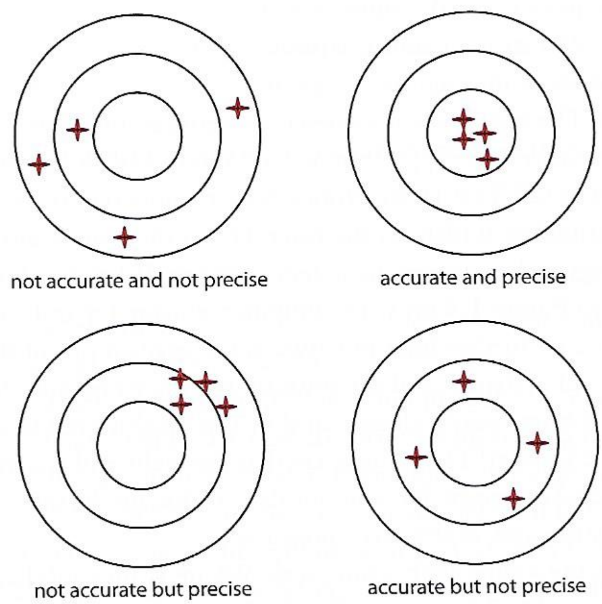


Figure 1.5 The meaning of accurate and precise measurements. Four different sets of four measurements each are shown.

Averages

In an experiment a measurement must be repeated many times, if at all possible. If it is repeated N times and the results of the measurements are $x_1, x_2, \ldots, x_N$, we calculate the **mean** or the **average** of these values ($\bar{x}$) using:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N}$$

This average is the **best estimate** for the quantity x based on the N measurements. What about the uncertainty? The best way is to get the **standard deviation** of the N numbers using your calculator. Standard deviation will not be examined but you may need to use it for your Internal Assessment, so it is good idea to learn it – you will learn it in your mathematics class anyway. The standard deviation σ of the N measurements is given by the formula (the calculator finds this very easily):

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2}{N-1}}$$

A very simple rule (not entirely satisfactory but acceptable for this course) is to use as an estimate of the uncertainty the quantity:

$$\Delta x = \frac{x_{\text{max}} - x_{\text{min}}}{2}$$

i.e. half of the difference between the largest and the smallest value.

For example, suppose we measure the period of a pendulum (in seconds) ten times:

1.20, 1.25, 1.30, 1.13, 1.25, 1.17, 1.41, 1.32, 1.29, 1.30

We calculate the mean:

$$\bar{t} = \frac{t_1 + t_2 + \cdots + t_{10}}{10} = 1.2620\,\text{s}$$

and the uncertainty:

$$\Delta t = \frac{t_{\text{max}} - t_{\text{min}}}{2} = \frac{1.41 - 1.13}{2} = 0.140\,\text{s}$$

How many significant figures do we use for uncertainties? The general rule is just one figure. So here we have $\Delta t = 0.1$ s. The uncertainty is in the first decimal place. **The value of the average period must also be expressed to the same precision as the uncertainty**, i.e. here to one decimal place, $\bar{t} = 1.3$ s. We then state that:

period = (1.3 ± 0.1) s

(Notice that each of the ten measurements of the period is subject to a reading error. Since these values were given to two decimal places, it is implied that the reading error is in the second decimal place, say ± 0.01 s.

Exam tip

There is some case to be made for using **two** significant figures in the uncertainty when the first digit in the uncertainty is 1. So in this example, since $\Delta t = 0.140$ s does begin with the digit 1, we should state $\Delta t = 0.14$ s and quote the result for the period as 'period = (1.26 ± 0.14) s'.

This is much smaller than the uncertainty found above so we ignore the reading error here. If instead the reading error were greater than the error due to the spread of values, we would have to include it instead. We will not deal with cases when the two errors are comparable.)

You will often see uncertainties with 2 s.f. in the scientific literature. For example, the charge of the electron is quoted as $e = (1.602\,176\,565 \pm 0.000\,000\,035) \times 10^{-19}$ C and the mass of the electron as $m_e = (9.109\,382\,91 \pm 0.000\,000\,40) \times 10^{-31}$ kg. This is perfectly all right and reflects the experimenter's **level of confidence** in his/her results. Expressing the uncertainty to 2 s.f. implies a more sophisticated statistical analysis of the data than is normally done in a high school physics course. With a lot of data, the measured values of e form a normal distribution with a given mean ($1.602\,176\,565 \times 10^{-19}$ C) and standard deviation ($0.000\,000\,035 \times 10^{-19}$ C). The experimenter is then 68% confident that the measured value of e lies within the interval [$1.602\,176\,530 \times 10^{-19}$ C, $1.602\,176\,600 \times 10^{-19}$ C].

Worked example

1.6 The diameter of a steel ball is to be measured using a micrometer caliper. The following are sources of error:

1 The ball is not centred between the jaws of the caliper.
2 The jaws of the caliper are tightened too much.
3 The temperature of the ball may change during the measurement.
4 The ball may not be perfectly round.

Determine which of these are random and which are systematic sources of error.

Sources 3 and 4 lead to unpredictable results, so they are random errors. Source 2 means that the measurement of diameter is always smaller since the calipers are tightened too much, so this is a systematic source of error. Source 1 certainly leads to unpredictable results depending on how the ball is centred, so it is a random source of error. But since the ball is not centred the 'diameter' measured is always smaller than the true diameter, so this is also a source of systematic error.

Propagation of uncertainties

A measurement of a length may be quoted as $L = (28.3 \pm 0.4)$ cm. The value 28.3 is called the **best estimate** or the **mean value** of the measurement and the 0.4 cm is called the **absolute uncertainty** in the measurement. The ratio of absolute uncertainty to mean value is called the **fractional uncertainty**. Multiplying the fractional uncertainty by 100% gives the **percentage uncertainty**. So, for $L = (28.3 \pm 0.4)$ cm we have that:

- absolute uncertainty = 0.4 cm
- fractional uncertainty $= \frac{0.4}{28.3} = 0.0141$
- percentage uncertainty $= 0.0141 \times 100\% = 1.41\%$

In general, if $a = a_0 \pm \Delta a$, we have:

- absolute uncertainty $= \Delta a$
- fractional uncertainty $= \frac{\Delta a}{a_0}$
- percentage uncertainty $= \frac{\Delta a}{a_0} \times 100\%$

The subscript 0 indicates the mean value, so a_0 is the mean value of a.

Suppose that three quantities are measured in an experiment: $a = a_0 \pm \Delta a$, $b = b_0 \pm \Delta b$, $c = c_0 \pm \Delta c$. We now wish to calculate a quantity Q in terms of a, b, c. For example, if a, b, c are the sides of a rectangular block we may want to find $Q = ab$, which is the area of the base, or $Q = 2a + 2b$, which is the perimeter of the base, or $Q = abc$, which is the volume of the block. Because of the uncertainties in a, b, c there will be an uncertainty in the calculated quantities as well. How do we calculate this uncertainty?

There are three cases to consider. We will give the results without proof.

Addition and subtraction

The first case involves the operations of addition and/or subtraction. For example, we might have $Q = a + b$ or $Q = a - b$ or $Q = a + b - c$. Then, **in all cases** the absolute uncertainty in Q is the **sum** of the **absolute** uncertainties in a, b and c.

$$Q = a + b \quad \Rightarrow \quad \Delta Q = \Delta a + \Delta b$$
$$Q = a - b \quad \Rightarrow \quad \Delta Q = \Delta a + \Delta b$$
$$Q = a + b - c \quad \Rightarrow \quad \Delta Q = \Delta a + \Delta b + \Delta c$$

Exam tip
In addition and subtraction, we always add the absolute uncertainties, never subtract.

Worked examples

1.7 The side a of a square, is measured to be (12.4 ± 0.1) cm. Find the perimeter P of the square including the uncertainty.

Because $P = a + a + a + a$, the perimeter is 49.6 cm. The absolute uncertainty in P is:

$$\Delta P = \Delta a + \Delta a + \Delta a + \Delta a$$

$$\Delta P = 4\Delta a$$

$$\Delta P = 0.4\,\text{cm}$$

Thus, $P = (49.6 \pm 0.4)$ cm.

1.8 Find the percentage uncertainty in the quantity $Q = a - b$, where $a = 538.7 \pm 0.3$ and $b = 537.3 \pm 0.5$. Comment on the answer.

The calculated value is 1.7 and the absolute uncertainty is $0.3 + 0.5 = 0.8$. So $Q = 1.4 \pm 0.8$.

The fractional uncertainty is $\frac{0.8}{1.4} = 0.57$, so the percentage uncertainty is 57%.

The fractional uncertainty in the quantities a and b is quite small. But the numbers are close to each other so their difference is very small. This makes the fractional uncertainty in the difference unacceptably large.

Multiplication and division

The second case involves the operations of multiplication and division. Here the **fractional uncertainty** of the result is the **sum** of the **fractional uncertainties** of the quantities involved:

$$Q = ab \quad \Rightarrow \quad \frac{\Delta Q}{Q_0} = \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0}$$

$$Q = \frac{a}{b} \quad \Rightarrow \quad \frac{\Delta Q}{Q_0} = \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0}$$

$$Q = \frac{ab}{c} \quad \Rightarrow \quad \frac{\Delta Q}{Q_0} = \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0} + \frac{\Delta c}{c_0}$$

Powers and roots

The third case involves calculations where quantities are raised to powers or roots. Here the **fractional uncertainty** of the result is the **fractional uncertainty** of the quantity **multiplied** by the absolute value of the power:

$$Q = a^n \quad \Rightarrow \quad \frac{\Delta Q}{Q_0} = |n| \frac{\Delta a}{a_0}$$

$$Q = \sqrt[n]{a} \quad \Rightarrow \quad \frac{\Delta Q}{Q_0} = \frac{1}{n} \frac{\Delta a}{a_0}$$

Worked examples

1.9 The sides of a rectangle are measured to be $a = 2.5\text{ cm} \pm 0.1\text{ cm}$ and $b = 5.0\text{ cm} \pm 0.1\text{ cm}$. Find the area A of the rectangle.

The fractional uncertainty in a is:

$$\frac{\Delta a}{a} = \frac{0.1}{2.5} = 0.04 \text{ or } 4\%$$

The fractional uncertainty in b is:

$$\frac{\Delta b}{b} = \frac{0.1}{5.0} = 0.02 \text{ or } 2\%$$

Thus, the fractional uncertainty in the area is $0.04 + 0.02 = 0.06$ or 6%.

The area A_0 is:

$$A_0 = 2.5 \times 5.0 = 12.5\text{ cm}^2$$

and $\frac{\Delta A}{A_0} = 0.06$

$\Rightarrow \quad \Delta A = 0.06 \times 12.5 = 0.75\text{ cm}^2$

Hence $A = 12.5\text{ cm}^2 \pm 0.8\text{ cm}^2$ (the final absolute uncertainty is quoted to 1 s.f.).

1.10 A mass is measured to be $m = 4.4 \pm 0.2\,\text{kg}$ and its speed v is measured to be $18 \pm 2\,\text{m s}^{-1}$. Find the kinetic energy of the mass.

The kinetic energy is $E = \frac{1}{2}mv^2$, so the mean value of the kinetic energy, E_0, is:

$$E_0 = \frac{1}{2} \times 4.4 \times 18^2 = 712.8\,\text{J}$$

Using:

$$\frac{\Delta E}{E_0} = \frac{\Delta m}{m_0} + \underbrace{2\times}_{\text{because of the square}} \frac{\Delta v}{v_0}$$

we find:

$$\frac{\Delta E}{712.8} = \frac{0.2}{4.4} = 2 \times \frac{2}{18} = 0.267$$

So:

$$\Delta E = 712.8 \times 0.2677 = 190.8\,\text{J}$$

To one significant figure, the uncertainty is $\Delta E = 200 = 2 \times 10^2\,\text{J}$; that is $E = (7 \pm 2) \times 10^2\,\text{J}$.

Exam tip

The final absolute uncertainty must be expressed to one significant figure. This limits the precision of the quoted value for energy.

1.11 The length of a simple pendulum is increased by 4%. What is the fractional increase in the pendulum's period?

The period T is related to the length L through $T = 2\pi\sqrt{\frac{L}{g}}$.

Because this relationship has a square root, the fractional uncertainties are related by:

$$\frac{\Delta T}{T_0} = \underbrace{\frac{1}{2}\times}_{\text{because of the square root}} \frac{\Delta L}{L_0}$$

We are told that $\frac{\Delta L}{L_0} = 4\%$. This means we have :

$$\frac{\Delta T}{T_0} = \frac{1}{2} \times 4\% = 2\%$$

1.12 A quantity Q is measured to be $Q=3.4\pm0.5$. Calculate the uncertainty in **a** $\frac{1}{Q}$ and **b** Q^2.

a $$\frac{1}{Q}=\frac{1}{3.4}=0.294\,118$$

$$\frac{\Delta(1/Q)}{1/Q}=\frac{\Delta Q}{Q}$$

$$\Rightarrow \quad \Delta(1/Q)=\frac{\Delta Q}{Q^2}=\frac{0.5}{3.4^2}=0.043\,25$$

Hence: $\frac{1}{Q}=0.29\pm0.04$

b $$Q^2=3.4^2=11.5600$$

$$\frac{\Delta(Q^2)}{Q^2}=2\times\frac{\Delta Q}{Q}$$

$$\Rightarrow \quad \Delta(Q^2)=2Q\times\Delta Q=2\times3.4\times0.5=3.4$$

Hence: $Q^2=12\pm3$

1.13 The volume of a cylinder of base radius r and height h is given by $V=\pi r^2 h$. The volume is measured with an uncertainty of 4% and the height with with an uncertainty of 2%. Determine the uncertainty in the radius.

We must first solve for the radius to get $r=\sqrt{\frac{V}{\pi h}}$. The uncertainty is then:

$$\frac{\Delta r}{r}\times100\% = \frac{1}{2}\left(\frac{\Delta V}{V}+\frac{\Delta h}{h}\right)\times100\% = \frac{1}{2}(4+2)\times100\% = 3\%$$

Best-fit lines

In mathematics, plotting a point on a set of axes is straightforward. In physics, it is slightly more involved because the point consists of measured or calculated values and so is subject to uncertainty. So the point $(x_0\pm\Delta x,\ y_0\pm\Delta y)$ is plotted as shown in Figure **1.6**. The uncertainties are

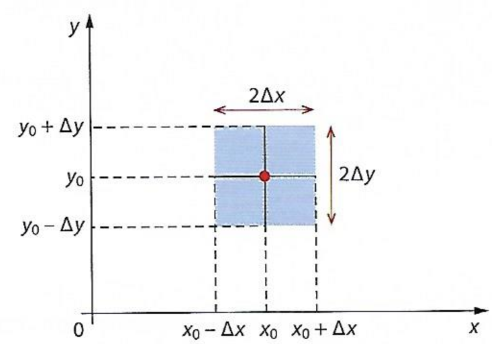


Figure 1.6 A point plotted along with its error bars.

represented by **error bars**. To 'go through the error bars' a best-fit line can go through the area shaded grey.

In a physics experiment we usually try to plot quantities that will give straight-line graphs. The graph in Figure **1.7** shows the variation with extension x of the tension T in a spring. The points and their error bars are plotted. The blue line is the best-fit line. It has been drawn by eye by trying to minimise the distance of the points from the line – this means that some points are above and some are below the best-fit line.

The gradient (slope) of the best-fit line is found by using two points **on the best-fit line** as far from each other as possible. We use (0, 0) and (0.0390, 7.88). The gradient is then:

$$\text{gradient} = \frac{\Delta F}{\Delta x}$$

$$\text{gradient} = \frac{7.88 - 0}{0.0390 - 0}$$

$$\text{gradient} = 202\,\text{N m}^{-1}$$

The best-fit line has equation $F = 202x$. (The vertical intercept is essentially zero; in this equation x is in metres and F in newtons.)

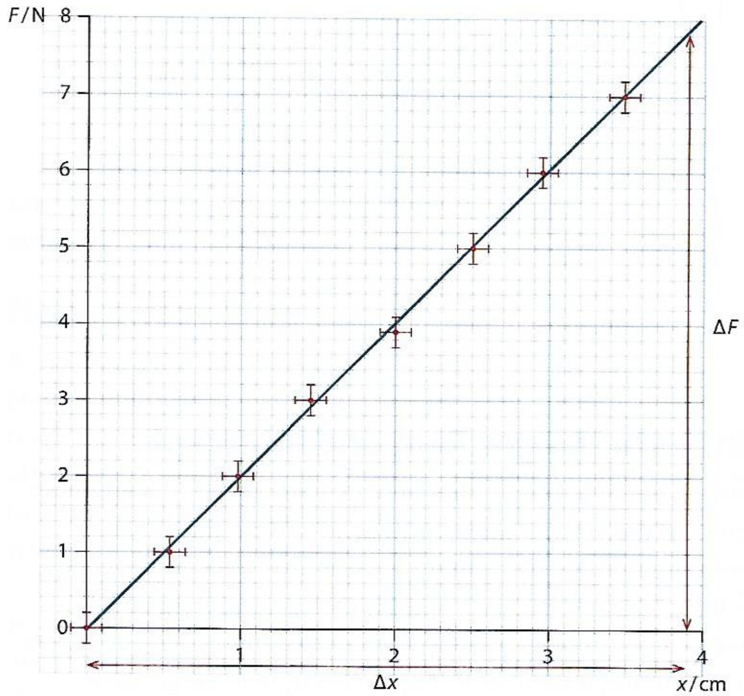


Figure 1.7 Data points plotted together with uncertainties in the values for the tension. To find the gradient, use two points on the best-fit line far apart from each other.

On the other hand it is perfectly possible to obtain data that cannot be easily manipulated to give a straight line. In that case a smooth curve passing through all the error bars is the best-fit line (Figure **1.8**).

From the graph the maximum power is 4.1 W, and it occurs when $R = 2.2\,\Omega$. The estimated uncertainty in R is about the length of a square, i.e. $\pm 0.1\,\Omega$. Similarly, for the power the estimated uncertainty is ± 0.1 W.

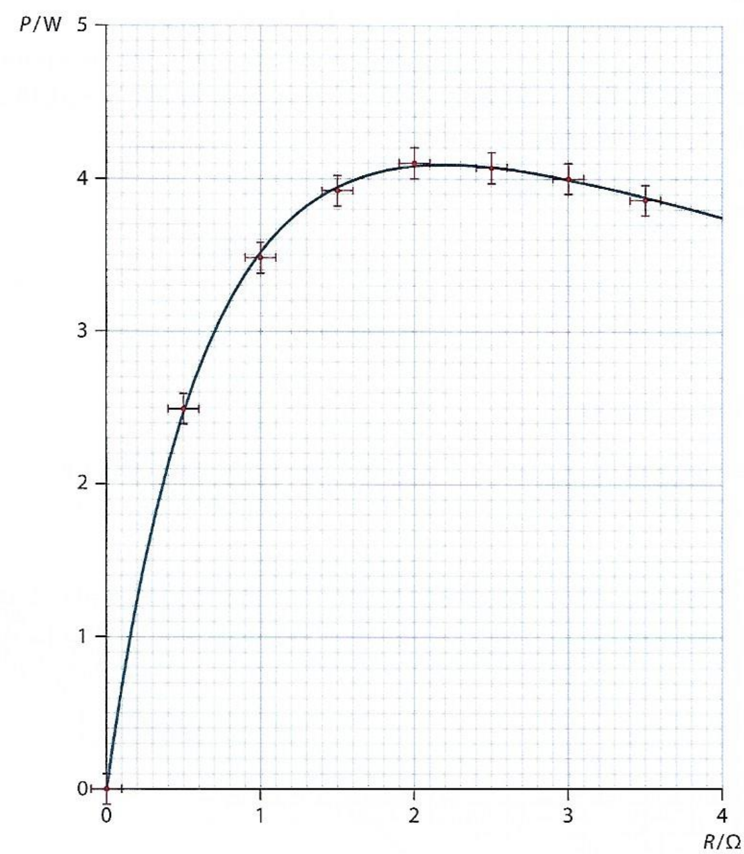


Figure 1.8 The best-fit line can be a curve.

Uncertainties in the gradient and intercept

When the best-fit line is a straight line we can easily obtain uncertainties in the gradient and the vertical intercept. The idea is to draw lines of maximum and minimum gradient in such a way that they go through all the **error bars** (not just the 'first' and the 'last' points). Figure **1.9** shows the best-fit line (in blue) and the lines of maximum and minimum gradient. The green line is the line through all error bars of greatest gradient. The red line is the line through all error bars with smallest gradient. All lines are drawn by eye.

The blue line has gradient $k_{max} = 210\,\mathrm{N\,m^{-1}}$ and intercept -0.18 N. The red line has gradient $k_{min} = 193\,\mathrm{N\,m^{-1}}$ and intercept $+0.13$ N. So we can find the uncertainty in the gradient as:

$$\Delta k = \frac{k_{max} - k_{min}}{2} = \frac{210 - 193}{2} = 8.5 \approx 8\,\mathrm{N\,m^{-1}}$$

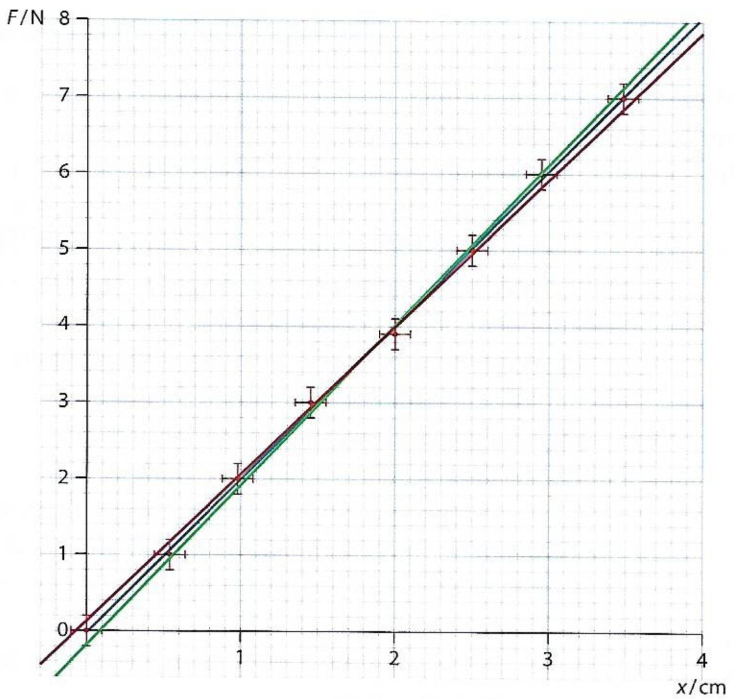


Figure 1.9 The best-fit line, along with lines of maximum and minimum gradient.

The uncertainty in the vertical intercept is similarly:

$$\Delta_{\text{intercept}} = \frac{0.13-(-0.18)}{2} = 0.155 \approx 0.2\,\text{N}$$

We saw earlier that the line of best fit has gradient $202\,\text{N}\,\text{m}^{-1}$ and zero intercept. So we quote the results as $k = (2.02 \pm 0.08) \times 10^2$ and intercept $= 0.0 \pm 0.2\,\text{N}$.

Nature of science

A key part of the scientific method is recognising the errors that are present in the experimental technique being used, and working to reduce these as much as possible. In this section you have learned how to calculate errors in quantities that are combined in different ways and how to estimate errors from graphs. You have also learned how to recognise systematic and random errors.

No matter how much care is taken, scientists know that their results are uncertain. But they need to distinguish between inaccuracy and uncertainty, and to know how confident they can be about the validity of their results. The search to gain more accurate results pushes scientists to try new ideas and refine their techniques. There is always the possibility that a new result may confirm a hypothesis for the present, or it may overturn current theory and open a new area of research. Being aware of doubt and uncertainty are key to driving science forward.

? Test yourself

23 The magnitudes of two forces are measured to be $120 \pm 5\,\text{N}$ and $60 \pm 3\,\text{N}$. Find the sum and difference of the two magnitudes, giving the uncertainty in each case.

24 The quantity Q depends on the measured values a and b in the following ways:

a $Q = \frac{a}{b}$, $a = 20 \pm 1$, $b = 10 \pm 1$

b $Q = 2a + 3b$, $a = 20 \pm 2$, $b = 15 \pm 3$

c $Q = a - 2b$, $a = 50 \pm 1$, $b = 24 \pm 1$

d $Q = a^2$, $a = 10.0 \pm 0.3$

e $Q = \frac{a^2}{b^2}$, $a = 100 \pm 5$, $b = 20 \pm 2$

In each case, find the value of Q and its uncertainty.

25 The centripetal force is given by $F = \frac{mv^2}{r}$. The mass is measured to be $2.8 \pm 0.1\,\text{kg}$, the velocity $14 \pm 2\,\text{m s}^{-1}$ and the radius $8.0 \pm 0.2\,\text{m}$; find the force on the mass, including the uncertainty.

26 The radius r of a circle is measured to be $2.4\,\text{cm} \pm 0.1\,\text{cm}$. Find the uncertainty in:

a the area of the circle

b the circumference of the circle.

27 The sides of a rectangle are measured as $4.4 \pm 0.2\,\text{cm}$ and $8.5 \pm 0.3\,\text{cm}$. Find the area and perimeter of the rectangle.

28 The length L of a pendulum is increased by 2%. Find the percentage increase in the period T.

$\left(T = 2\pi\sqrt{\frac{L}{g}}\right)$

29 The volume of a cone of base radius R and height h is given by $V = \frac{\pi R^2 h}{3}$. The uncertainty in the radius and in the height is 4%. Find the percentage uncertainty in the volume.

30 In an experiment to measure current and voltage across a device, the following data was collected: $(V, I) = \{(0.1, 26), (0.2, 48), (0.3, 65), (0.4, 90)\}$. The current was measured in mA and the voltage in mV. The uncertainty in the current was $\pm 4\,\text{mA}$. Plot the current versus the voltage and draw the best-fit line through the points. Suggest whether the current is proportional to the voltage.

31 In a similar experiment to that in question **30**, the following data was collected for current and voltage: $(V, I) = \{(0.1, 27), (0.2, 44), (0.3, 60), (0.4, 78)\}$ with an uncertainty of $\pm 4\,\text{mA}$ in the current. Plot the current versus the voltage and draw the best-fit line. Suggest whether the current is proportional to the voltage.

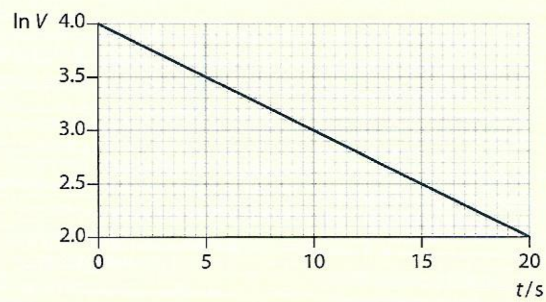
32 A circle and a square have the same perimeter. Which shape has the larger area?

33 The graph shows the natural logarithm of the voltage across a capacitor of capacitance $C = 5.0\,\mu\text{F}$ as a function of time. The voltage is given by the equation $V = V_0 e^{-t/RC}$, where R is the resistance of the circuit. Find:

a the initial voltage

b the time for the voltage to be reduced to half its initial value

c the resistance of the circuit.



34 The table shows the mass M of several stars and their corresponding luminosity L (power emitted).

a Plot L against M and draw the best-fit line.

b Plot the logarithm of L against the logarithm of M. Use your graph to find the relationship between these quantities, assuming a power law of the kind $L = kM^{\alpha}$. Give the numerical value of the parameter α.

Mass *M* (in solar masses)	**Luminosity *L* (in terms of the Sun's luminosity)**
1.0 ± 0.1	1 ± 0
3.0 ± 0.3	42 ± 4
5.0 ± 0.5	230 ± 20
12 ± 1	4700 ± 50
20 ± 2	$26\,500 \pm 300$

1.3 Vectors and scalars

Learning objectives

- Distinguish between vector and scalar quantities.
- Resolve a vector into its components.
- Reconstruct a vector from its components.
- Carry out operations with vectors.

Quantities in physics are either scalars (i.e. they just have magnitude) or vectors (i.e. they have magnitude and direction). This section provides the tools you need for dealing with vectors.

Vectors

Some quantities in physics, such as time, distance, mass, speed and temperature, just need one number to specify them. These are called **scalar** quantities. For example, it is sufficient to say that the mass of a body is 64 kg or that the temperature is −5.0 °C. On the other hand, many quantities are fully specified only if, in addition to a number, a direction is needed. Saying that you will leave Paris now, in a train moving at 220 km/h, does not tell us where you will be in 30 minutes because we do not know the **direction** in which you will travel. Quantities that need a direction in addition to magnitude are called **vector** quantities. Table **1.7** gives some examples of vector and scalars.

A vector is represented by a straight arrow, as shown in Figure **1.10a**. The direction of the arrow represents the direction of the vector and the length of the arrow represents the **magnitude** of the vector. To say that two vectors are the same means that **both** magnitude and direction are the same. The vectors in Figure **1.10b** are all equal to each other. In other words, vectors do not have to start from the same point to be equal.

We write vectors as italic boldface $\boldsymbol{a}$. The magnitude is written as $|\boldsymbol{a}|$ or just a.

Vectors	Scalars
displacement	distance
velocity	speed
acceleration	mass
force	time
weight	density
electric field	electric potential
magnetic field	electric charge
gravitational field	gravitational potential
momentum	temperature
area	volume
angular velocity	work/energy/power

Table 1.7 Examples of vectors and scalars.

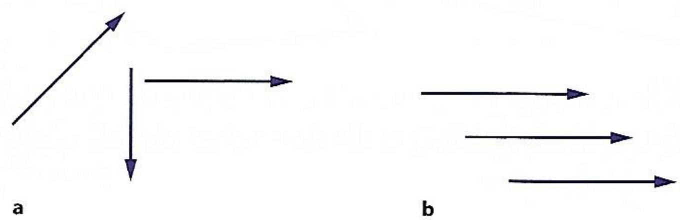


Figure 1.10 **a** Representation of vectors by arrows. **b** These three vectors are equal to each other.

Multiplication of a vector by a scalar

A vector can be multiplied by a number. The vector $\boldsymbol{a}$ multiplied by the number 2 gives a vector in the same direction as $\boldsymbol{a}$ but 2 times longer. The vector $\boldsymbol{a}$ multiplied by −0.5 is opposite to $\boldsymbol{a}$ in direction and half as long (Figure **1.11**). The vector $-\boldsymbol{a}$ has the same magnitude as $\boldsymbol{a}$ but is opposite in direction.

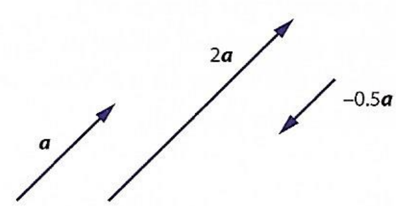


Figure1.11 Multiplication of vectors by a scalar.

Addition of vectors

Figure **1.12a** shows vectors ***d*** and ***e***. We want to find the vector that equals ***d*** + ***e***. Figure **1.12b** shows one method of adding two vectors.

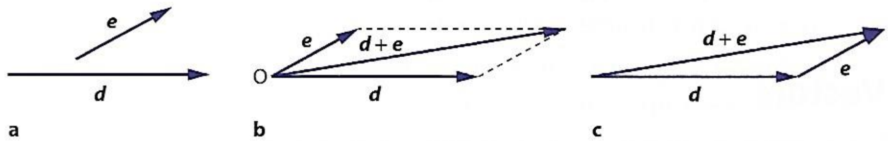


Figure 1.12 **a** Vectors **d** and **e**. **b** Adding two vectors involves shifting one of them parallel to itself so as to form a parallelogram with the two vectors as the two sides. The diagonal represents the sum. **c** An equivalent way to add vectors.

To add two vectors:

1 Draw them so they start at a common point O.
2 Complete the parallelogram whose sides are ***d*** and ***e***.
3 Draw the diagonal of this parallelogram starting at O. This is the vector ***d*** + ***e***.

Equivalently, you can draw the vector ***e*** so that it starts where the vector ***d*** stops and then join the beginning of ***d*** to the end of ***e***, as shown in Figure **1.12c**.

Exam tip

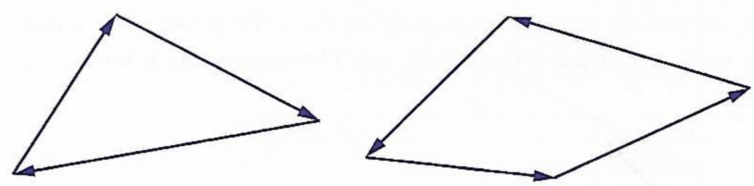


Figure 1.13

Vectors (with arrows pointing in the same sense) forming closed polygons add up to zero.

Exam tip

The change in a quantity, and in particular the change in a vector quantity, will follow us through this entire course. You need to learn this well.

Subtraction of vectors

Figure **1.14** shows vectors ***d*** and ***e***. We want to find the vector that equals ***d*** − ***e***.

To subtract two vectors:

1 Draw them so they start at a common point O.
2 The vector from the tip of ***e*** to the tip of ***d*** is the vector ***d*** − ***e***.

(Notice that is equivalent to adding ***d*** to −***e***.)

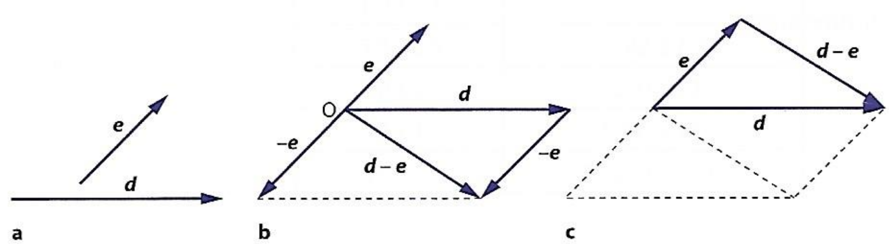


Figure 1.14 Subtraction of vectors.

Worked examples

1.14 Copy the diagram in Figure **1.15a**. Use the diagram to draw the third force that will keep the point P in equilibrium.

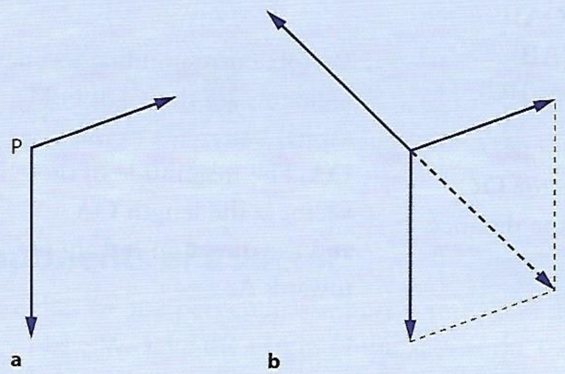


Figure 1.15

We find the sum of the two given forces using the parallelogram rule and then draw the opposite of that vector, as shown in Figure **1.15b**.

1.15 A velocity vector of magnitude $1.2\,\mathrm{m\,s^{-1}}$ is horizontal. A second velocity vector of magnitude $2.0\,\mathrm{m\,s^{-1}}$ must be added to the first so that the sum is vertical in direction. Find the direction of the second vector and the magnitude of the sum of the two vectors.

We need to draw a scale diagram, as shown in Figure **1.16**. Representing $1.0\,\mathrm{m\,s^{-1}}$ by 2.0 cm, we see that the $1.2\,\mathrm{m\,s^{-1}}$ corresponds to 2.4 cm and $2.0\,\mathrm{m\,s^{-1}}$ to 4.0 cm.

First draw the horizontal vector. Then mark the vertical direction from O. Using a compass (or a ruler), mark a distance of 4.0 cm from A, which intersects the vertical line at B. AB must be one of the sides of the parallelogram we are looking for.

Now measure a distance of 2.4 cm horizontally from B to C and join O to C. This is the direction in which the second velocity vector must be pointing. Measuring the diagonal OB (i.e. the vector representing the sum), we find 3.2 cm, which represents $1.6\,\mathrm{m\,s^{-1}}$. Using a protractor, we find that the $2.0\,\mathrm{m\,s^{-1}}$ velocity vector makes an angle of about 37° with the vertical.

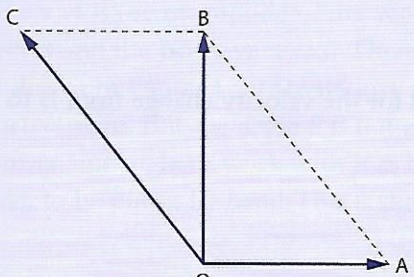


Figure 1.16 Using a scale diagram to solve a vector problem.

1.16 A person walks 5.0 km east, followed by 3.0 km north and then another 4.0 km east. Find their final position.

The walk consists of three steps. We may represent each one by a vector (Figure **1.17**).

- The first step is a vector of magnitude 5.0 km directed east (**OA**).
- The second is a vector of magnitude 3.0 km directed north (**AB**).
- The last step is represented by a vector of 4.0 km directed east (**BC**).

The person will end up at a place that is given by the vector sum of these three vectors, that is **OA** + **AB** + **BC**, which equals the vector **OC**. By measurement from a scale drawing, or by simple geometry, the distance from O to C is 9.5 km and the angle to the horizontal is 18.4°.

Vectors corresponding to line segments are shown as bold capital letters, for example **OA**. The magnitude of the vector is the length OA and the direction is from O towards A.

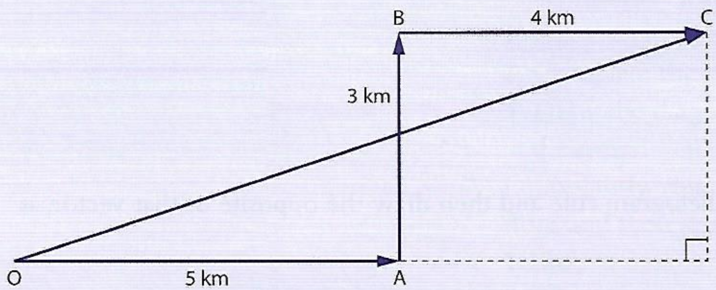


Figure 1.17 Scale drawing using 1 cm = 1 km.

1.17 A body moves in a circle of radius 3.0 m with a constant speed of 6.0 m s^{-1}. The velocity vector is at all times tangent to the circle. The body starts at A, proceeds to B and then to C. Find the change in the velocity vector between A and B and between B and C (Figure **1.18**).

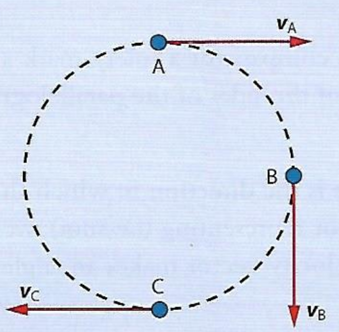


Figure 1.18

For the velocity change from A to B we have to find the difference $\boldsymbol{v}_B - \boldsymbol{v}_A$. and for the velocity change from B to C we need to find $\boldsymbol{v}_C - \boldsymbol{v}_B$. The vectors are shown in Figure **1.19**.

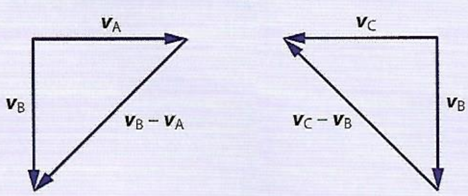


Figure 1.19

The vector $v_B - v_A$ is directed south-west and its magnitude is (by the Pythagorean theorem):

$$\sqrt{v_A{}^2 + v_B{}^2} = \sqrt{6^2 + 6^2}$$
$$= \sqrt{72}$$
$$= 8.49\,\text{m s}^{-1}$$

The vector $v_C - v_B$ has the same magnitude as $v_B - v_A$ but is directed north-west.

Components of a vector

Suppose that we use perpendicular axes x and y and draw vectors on this x–y plane. We take the origin of the axes as the starting point of the vector. (Other vectors whose beginning points are not at the origin can be shifted parallel to themselves until they, too, begin at the origin.) Given a vector $\boldsymbol{a}$ we define its **components along the axes** as follows. From the tip of the vector draw lines parallel to the axes and mark the point on each axis where the lines intersect the axes (Figure **1.20**).

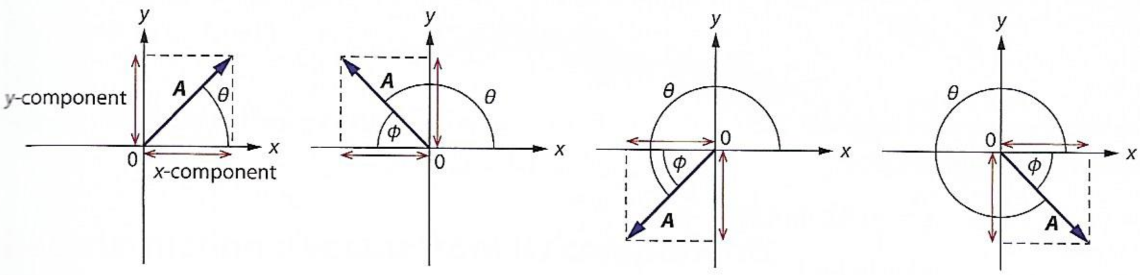


Figure 1.20 The components of a vector ***A*** and the angle needed to calculate the components. The angle θ is measured counter-clockwise from the positive x-axis.

The x- and y-components of A are called A_x and A_y. They are given by:

$$A_x = A\cos\theta$$

$$A_y = A\sin\theta$$

where A is the magnitude of the vector and θ is the angle between the vector and the **positive** x-axis. These formulas and the angle θ defined as shown in Figure **1.20** always give the correct components with the correct signs. But the angle θ is not always the most convenient. A more convenient angle to work with is φ, but when using this angle the signs have to be put in by hand. This is shown in Worked example **1.18**.

Exam tip
The formulas given for the components of a vector can **always** be used, but the angle must be the one defined in Figure **1.20**, which is sometimes awkward. You can use other more convenient angles, but then the formulas for the components may change.

Worked examples

1.18 Find the components of the vectors in Figure **1.21**. The magnitude of ***a*** is 12.0 units and that of ***b*** is 24.0 units.

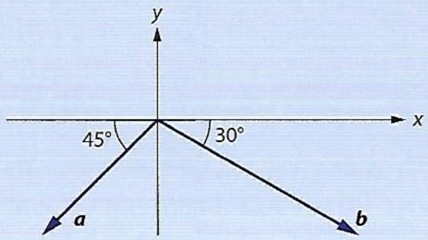


Figure 1.21

Taking the angle from the positive x-axis, the angle for ***a*** is $\theta = 180° + 45° = 225°$ and that for ***b*** is $\theta = 270° + 60° = 330°$. Thus:

$a_x = 12.0 \cos 225°$ $\quad$ $b_x = 24.0 \cos 330°$

$a_x = -8.49$ $\quad$ $b_x = 20.8$

$a_y = 12.0 \sin 225°$ $\quad$ $b_y = 24.0 \sin 330°$

$a_y = -8.49$ $\quad$ $b_y = -12.0$

But we do not have to use the awkward angles of 225° and 330°. For vector ***a*** it is better to use the angle of $\varphi = 45°$. In that case simple trigonometry gives:

$a_x = -12.0 \cos 45° = -8.49$ and $a_y = -12.0 \sin 45° = -8.49$

↑ put in by hand $\quad$ ↑ put in by hand

For vector ***b*** it is convenient to use the angle of $\varphi = 30°$, which is the angle the vector makes with the x-axis. But in this case:

$b_x = 24.0 \cos 30° = 20.8$ and $b_y = -24.0 \sin 30° = -12.0$

↑ put in by hand

1.19 Find the components of the vector W along the axes shown in Figure **1.22**.

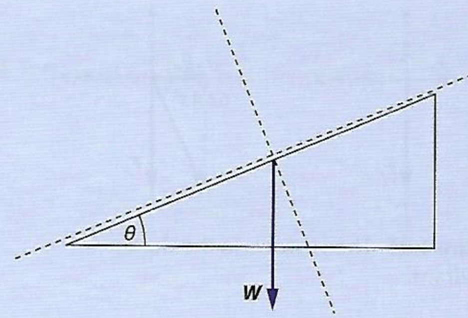


Figure 1.22

See Figure **1.23**. Notice that the angle between the vector W and the negative y-axis is θ.

Then by simple trigonometry

$W_x = -W\sin\theta$ (W_x is opposite the angle θ so the sine is used)

$W_y = -W\cos\theta$ (W_y is adjacent to the angle θ so the cosine is used)

(Both components are along the negative axes, so a minus sign has been put in by hand.)

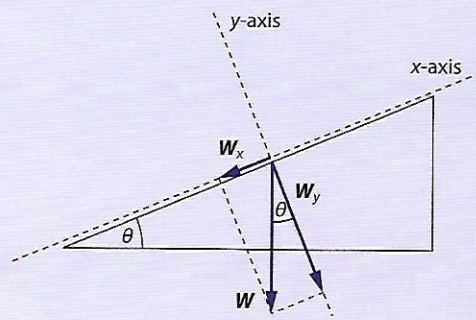


Figure 1.23

Reconstructing a vector from its components

Knowing the components of a vector allows us to reconstruct it (i.e. to find the magnitude and direction of the vector). Suppose that we are given that the x- and y-components of a vector are F_x and F_y. We need to find the magnitude of the vector F and the angle (θ) it makes with the x-axis (Figure **1.24**). The magnitude is found by using the Pythagorean theorem and the angle by using the definition of tangent.

$$F = \sqrt{F_x^2 + F_y^2}, \quad \theta = \arctan\frac{F_y}{F_x}$$

As an example, consider the vector whose components are $F_x = 4.0$ and $F_y = 3.0$. The magnitude of $\boldsymbol{F}$ is:

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4.0^2 + 3.0^2} = \sqrt{25} = 5.0$$

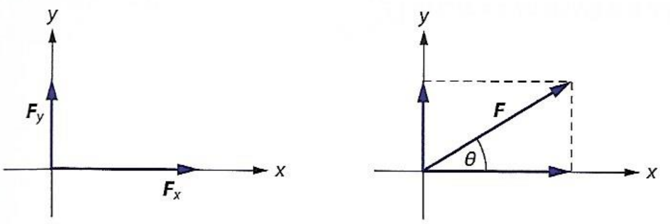


Figure 1.24 Given the components of a vector we can find its magnitude and direction.

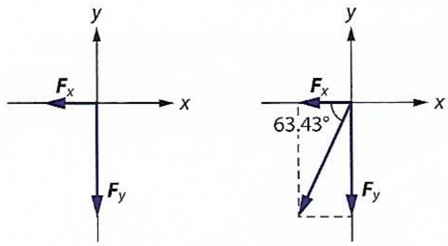


Figure 1.25 The vector is in the third quadrant.

and the direction is found from:

$$\theta = \arctan\frac{F_y}{F_x} = \arctan\frac{3}{4} = 36.87° \approx 37°$$

Here is another example. We need to find the magnitude and direction of the vector with components $F_x = -2.0$ and $F_y = -4.0$. The vector lies in the third quadrant, as shown in Figure **1.25**.

The magnitude is:

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(-2.0)^2 + (-4.0)^2}$$

$$= \sqrt{20} = 4.47 \approx 4.5$$

The direction is found from:

$$\varphi = \arctan\frac{F_y}{F_x} = \arctan\frac{-4}{-2} = \arctan 2$$

The calculator gives $\theta = \tan^{-1} 2 = 63°$. This angle is the one shown in Figure **1.25**.

In general, the simplest procedure to find the angle without getting stuck in trigonometry is to evaluate $\varphi = \arctan\left|\frac{F_y}{F_x}\right|$ i.e. **ignore the signs** in the components. The calculator will then give you the angle between the vector and the x-axis, as shown in Figure **1.26**.

Adding or subtracting vectors is very easy when we have the components, as Worked example **1.20** shows.

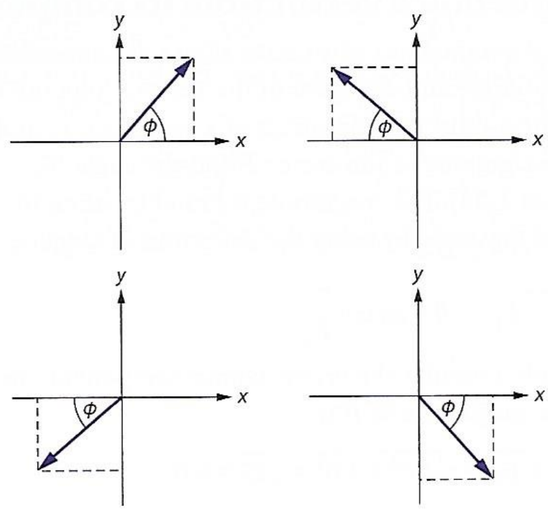


Figure 1.26 The angle φ is given by $\varphi = \arctan\left|\frac{F_y}{F_x}\right|$

Worked example

1.20 Find the sum of the vectors shown in Figure **1.27**. $\boldsymbol{F}_1$ has magnitude 8.0 units and $\boldsymbol{F}_2$ has magnitude 12 units. Their directions are as shown in the diagram.

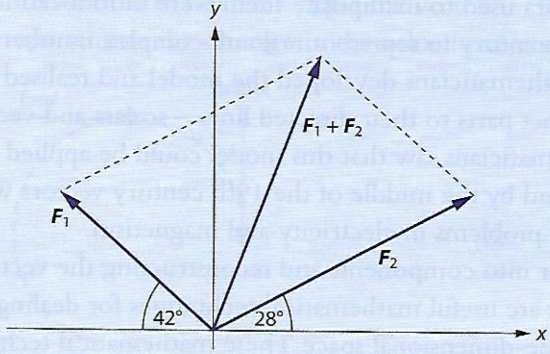


Figure 1.27 The sum of vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ (not to scale).

Find the components of the two vectors:

$F_{1x} = -F_1 \cos 42°$

$F_{1x} = -5.945$

$F_{1y} = F_1 \sin 42°$

$F_{1y} = 5.353$

$F_{2x} = F_2 \cos 28°$

$F_{2x} = 10.595$

$F_{2y} = F_2 \sin 28°$

$F_{2y} = 5.634$

The sum $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$ then has components:

$F_x = F_{1x} + F_{2x} = 4.650$

$F_y = F_{1y} + F_{2y} = 10.987$

The magnitude of the sum is therefore:

$F = \sqrt{4.650^2 + 10.987^2}$

$F = 11.9 \approx 12$

and its direction is:

$\varphi = \arctan\left(\frac{10.987}{4.65}\right)$

$\varphi = 67.1 \approx 67°$

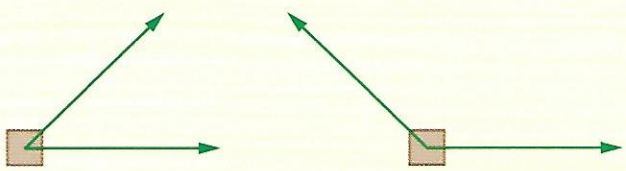
Nature of science

For thousands of years, people across the world have used maps to navigate from one place to another, making use of the ideas of distance and direction to show the relative positions of places. The concept of vectors and the algebra used to manipulate them were introduced in the first half of the 19th century to represent real and complex numbers in a geometrical way. Mathematicians developed the model and realised that there were two distinct parts to their directed lines – scalars and vectors. Scientists and mathematicians saw that this model could be applied to theoretical physics, and by the middle of the 19th century vectors were being used to model problems in electricity and magnetism.

Resolving a vector into components and reconstructing the vector from its components are useful mathematical techniques for dealing with measurements in three-dimensional space. These mathematical techniques are invaluable when dealing with physical quantities that have both magnitude and direction, such as calculating the effect of multiple forces on an object. In this section you have done this in two dimensions, but vector algebra can be applied to three dimensions and more.

? Test yourself

35 A body is acted upon by the two forces shown in the diagram. In each case draw the one force whose effect on the body is the same as the two together.



36 Vector $\boldsymbol{A}$ has a magnitude of 12.0 units and makes an angle of 30° with the positive x-axis. Vector $\boldsymbol{B}$ has a magnitude of 8.00 units and makes an angle of 80° with the positive x-axis. Using a graphical method, find the magnitude and direction of the vectors:

a $A+B$ **b** $A-B$ **c** $A-2B$

37 Repeat the previous problem, this time using components.

38 Find the magnitude and direction of the vectors with components:

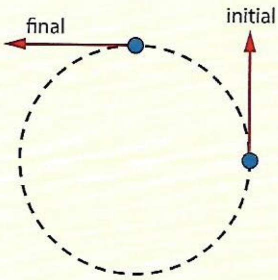
a $A_x=-4.0\,\text{cm}$, $A_y=-4.0\,\text{cm}$
b $A_x=124\,\text{km}$, $A_y=-158\,\text{km}$
c $A_x=0$, $A_y=-5.0\,\text{m}$
d $A_x=8.0\,\text{N}$, $A_y=0$

39 The components of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are as follows: ($A_x=2.00$, $A_y=3.00$), ($B_x=-2.00$, $B_y=5.00$). Find the magnitude and direction of the vectors:

a A **b** B **c** $A+B$
d $A-B$ **e** $2A-B$

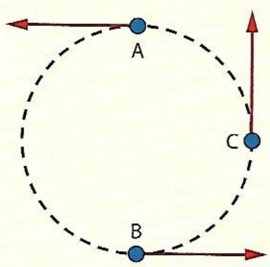
40 The position vector of a moving object has components ($r_x=2$, $r_y=2$) initially. After a certain time the position vector has components ($r_x=4$, $r_y=8$). Find the displacement vector.

41 The diagram shows the velocity vector of a particle moving in a circle with speed $10\,\mathrm{m\,s^{-1}}$ at two separate points. The velocity vector is tangential to the circle. Find the vector representing the **change** in the velocity vector.



42 In a certain collision, the momentum vector of a particle changes direction but not magnitude. Let $\boldsymbol{p}$ be the momentum vector of a particle suffering an elastic collision and changing direction by 30°. Find, in terms of p ($= |\boldsymbol{p}|$), the magnitude of the vector representing the change in the momentum vector.

43 The velocity vector of an object moving on a circular path has a direction that is tangent to the path (see diagram).



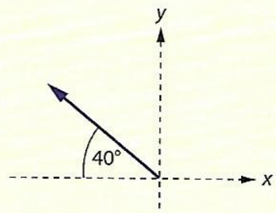
If the speed (magnitude of velocity) is constant at $4.0\,\mathrm{m\,s^{-1}}$, find the change in the velocity vector as the object moves:

a from A to B

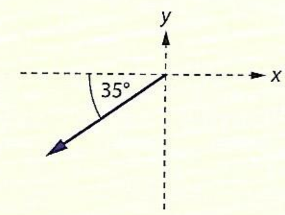
b from B to C.

c What is the change in the velocity vector from A to C? How is this related to your answers to **a** and **b**?

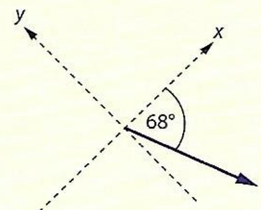
44 For each diagram, find the components of the vectors along the axes shown. Take the magnitude of each vector to be 10.0 units.



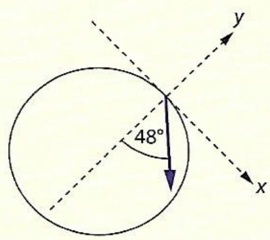
A



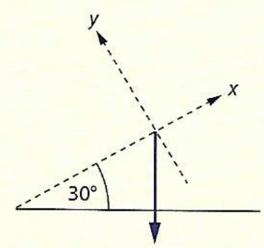
B



C



D



E

45 Vector **A** has a magnitude of 6.00 units and is directed at 60° to the positive x-axis. Vector **B** has a magnitude of 6.00 units and is directed at 120° to the positive x-axis. Find the magnitude and direction of vector **C** such that $A+B+C=0$. Place the three vectors so that one begins where the previous ends. What do you observe?

46 Plot the following pairs of vectors on a set of x- and y-axes. The angles given are measured counter-clockwise from the positive x-axis. Then, using the algebraic component method, find their sum in magnitude and direction.

a 12.0 N at 20° and 14.0 N at 50°

b 15.0 N at 15° and 18.0 N at 105°

c 20.0 N at 40° and 15.0 N at 310° (i.e. −50°)