Mechanics 2

2.1 Motion

This section is an introduction to the basic concepts used in describing motion. We will begin with motion in a straight line with constant velocity and then constant acceleration. Knowledge of uniformly accelerated motion allows analysis of more complicated motions, such as the motion of projectiles.

Learning objectives

- Understand the difference between distance and displacement.
- Understand the difference between speed and velocity.
- Understand the concept of acceleration.
- Analyse graphs describing motion.
- Solve motion problems using the equations for constant acceleration.
- Discuss the motion of a projectile.
- Show a qualitative understanding of the effects of a fluid resistance force on motion.
- Understand the concept of terminal speed.

Kinematical quantities

We will begin our discussion of motion with straight line motion in one dimension. This means that the particle that moves is constrained to move along a straight line. The **position** of the particle is then described by its coordinate on the straight line (Figure **2.1a**). If the line is horizontal, we may use the symbol x to represent the coordinate and hence the position. If the line is vertical, the symbol y is more convenient. In general, for an arbitrary line we may use a generic name, s, for position. So in Figure **2.1**, $x = 6\,\text{m}$, $y = -4\,\text{m}$ and $s = 0$.

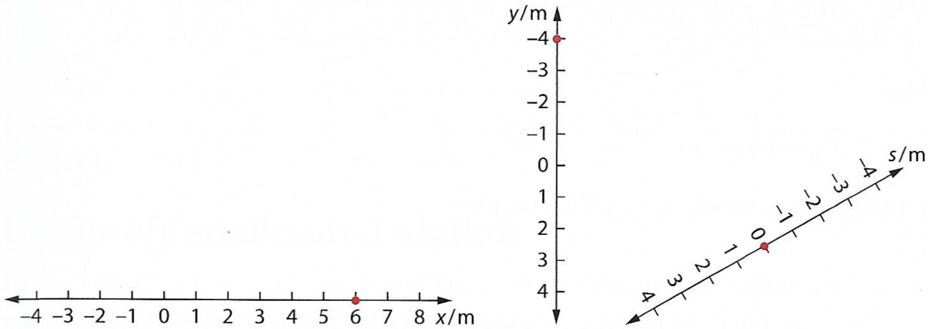


Figure 2.1 The position of a particle is determined by the coordinate on the number line.

As the particle moves on the straight line its position changes. In **uniform motion** the graph of position against time is a straight line (Figure **2.2**). In equal intervals of time, the position changes by the same amount. This means that the slope of the position–time graph is constant. This slope is defined to be the **average velocity** of the particle:

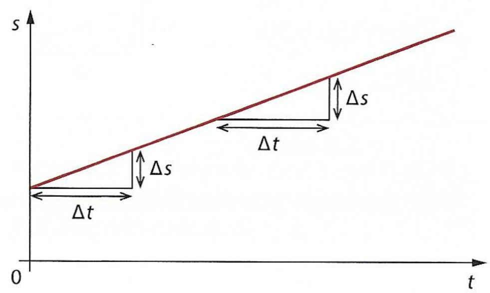


Figure 2.2 In uniform motion the graph of position versus time is a straight line.

$$v = \frac{\Delta s}{\Delta t}$$

where Δs is the change in position.

> The average velocity during an interval of time Δt is the ratio of the change in position Δs during that time interval to Δt.

(In uniform motion velocity is constant so the term 'average' is unnecessary. The velocity is the same at all times.)

Positive velocity means that the coordinate s that gives the position is increasing. Negative velocity means that s is decreasing.

Suppose we choose a time interval from $t=0$ to some arbitrary time t later. Let the position at $t=0$ (the initial position) be s_i and the position at time t be s. Then:

$$v=\frac{s-s_i}{t-0}$$

which can be re-arranged to give:

$$s=s_i+vt$$

This formula gives, in uniform motion, the position s of the moving object t seconds after time zero, given that the velocity is v and the initial position is s_i.

Worked example

2.1 Two cyclists, A and B, start moving at the same time. The initial position of A is 0 m and her velocity is $+20\,\text{km h}^{-1}$. The initial position of B is 150 km and he cycles at a velocity of $-30\,\text{km h}^{-1}$. Determine the time and position at which they will meet.

The position of A is given by the formula: $s_A=0+20t$

The position of B is given by the formula: $s_B=150-30t$

They will meet when they are the same position, i.e. when $s_A=s_B$. This implies:

$$20t=150-30t$$

$$50t=150$$

$$t=3.0\text{ hours}$$

The common position is found from either $s_A=20\times3.0=60\,\text{km}$ or $s_B=150-30\times3.0=60\,\text{km}$.

Consider two motions shown in Figure **2.3**. In the first, the particle leaves its initial position s_i at -4 m and continues to its final position at 16 m. The change in position is called **displacement** and in this case equals $16-(-4)=20$ m. The **distance** travelled is the actual length of the path followed and in this case is also 20 m.

Displacement = change in position
Distance = length of path followed

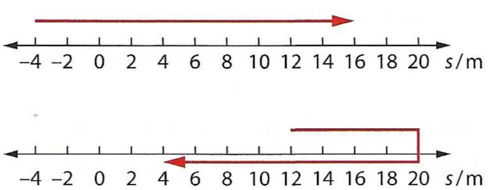


Figure 2.3 A motion in which the particle changes direction.

In the second motion, the particle leaves its initial position at 12 m, arrives at position 20 m and then comes back to its final position at 4.0 m.

The second motion is an example of motion with changing direction. The change in the position of this particle, i.e. the displacement is $\Delta s = s_f - s_i = 4.0 - 12 = -8.0\,\text{m}$. But the distance travelled by the particle (the length of the path) is 8.0 m in the outward trip and 16 m on the return trip, making a total distance of 24 m. So we must be careful to distinguish distance from displacement. Distance is a scalar but displacement is a vector. Numerically, they are different if there is a change of direction, as in this example.

For constant velocity, the graph of velocity versus time gives a horizontal straight line (Figure **2.4a**). An example of this type of motion is coasting in a straight line on a bicycle on level ground (Figure **2.4b**).

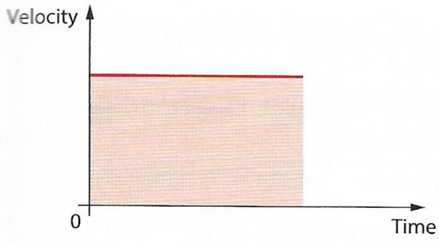


Figure 2.4 a In uniform motion the graph of velocity versus time is a horizontal straight line. **b** This motion is a good approximation to uniform motion.

But we now observe that the area under the graph from $t = 0$ to time t is vt. From $s = s_i + vt$ we deduce that this area is the change in position or the displacement.

Uniformly accelerated motion

In the last section we discussed uniform motion. This means motion in a straight line with **constant velocity**. In such motion the graph of position versus time is a straight line.

In most motions velocity is not constant. In **uniformly accelerated motion** the graph of velocity versus time is a non-horizontal straight line (Figure **2.5**).

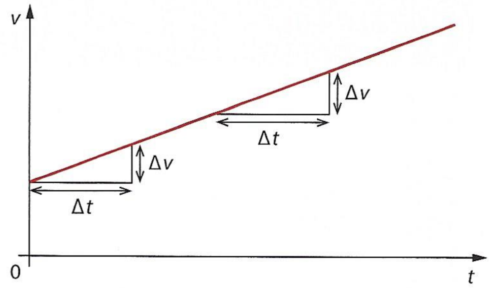


Figure 2.5 In uniformly accelerated motion the graph of velocity versus time is a straight line with non-zero slope.

In equal intervals of time the velocity changes by the same amount. The slope of the velocity–time graph is constant. This slope is defined to be the acceleration of the particle:

$$a = \frac{\Delta v}{\Delta t}$$

> **Acceleration** is the rate of change of velocity.

When the acceleration is positive, the velocity is increasing (Figure **2.6**). Negative acceleration means that v is decreasing. The plane reaches a take-off speed of $260\,\text{km}\,\text{h}^{-1}$ (about $72\,\text{m}\,\text{s}^{-1}$) in about 2 seconds, implying an average acceleration of about $36\,\text{m}\,\text{s}^{-2}$. The distance travelled until take-off is about 72 m.

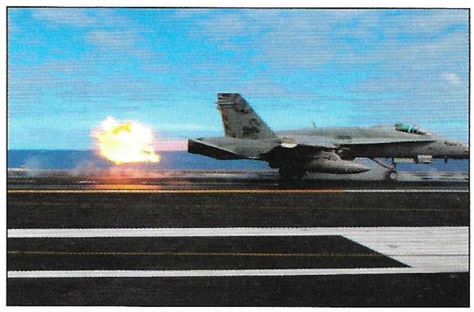


Figure 2.6 This F/ A-18C is accelerating!

Suppose we choose a time interval from $t=0$ to some arbitrary time t later. Let the velocity at $t=0$ (the initial velocity) be u and the velocity at time t be v. Then:

$$a=\frac{v-u}{t-0}$$

which can be re-arranged to:

$$v=u+at$$

For uniformly accelerated motion, this formula gives the velocity v of the moving object t seconds after time zero, given that the initial velocity is u and the acceleration is a.

Worked example

2.2 A particle has initial velocity $12\,\mathrm{m\,s^{-1}}$ and moves with a constant acceleration of $-3.0\,\mathrm{m\,s^{-2}}$. Determine the time at which the particle stops instantaneously.

The particle is getting slower. At some point it will stop instantaneously, i.e. its velocity v will be zero.

We know that $v=u+at$. Just substituting values gives:

$$0=12+(-3.0)\times t$$

$$3.0t=12$$

Hence $t=4.0\,\mathrm{s}$.

Defining velocity in non-uniform motion

But how is velocity defined now that it is not constant? We define the average velocity as before:

$$\bar{v}=\frac{\Delta s}{\Delta t}$$

But since the velocity changes, it has different values at different times. We would like to have a concept of the velocity at an instant of time, the **instantaneous velocity**. We need to make the time interval Δt very small. The instantaneous velocity is then defined as:

$$v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

In other words, instantaneous velocity is the average velocity obtained during an interval of time that is very, very small. In calculus, we learn that $\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$ has the following meaning: look at the graph of position s versus time t shown in Figure **2.7a**. As there is uniform acceleration, the graph is a curve. Choose a point on this curve. Draw the tangent line to the curve at the point. The slope of the tangent line is the meaning of $\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$ and therefore also of velocity.

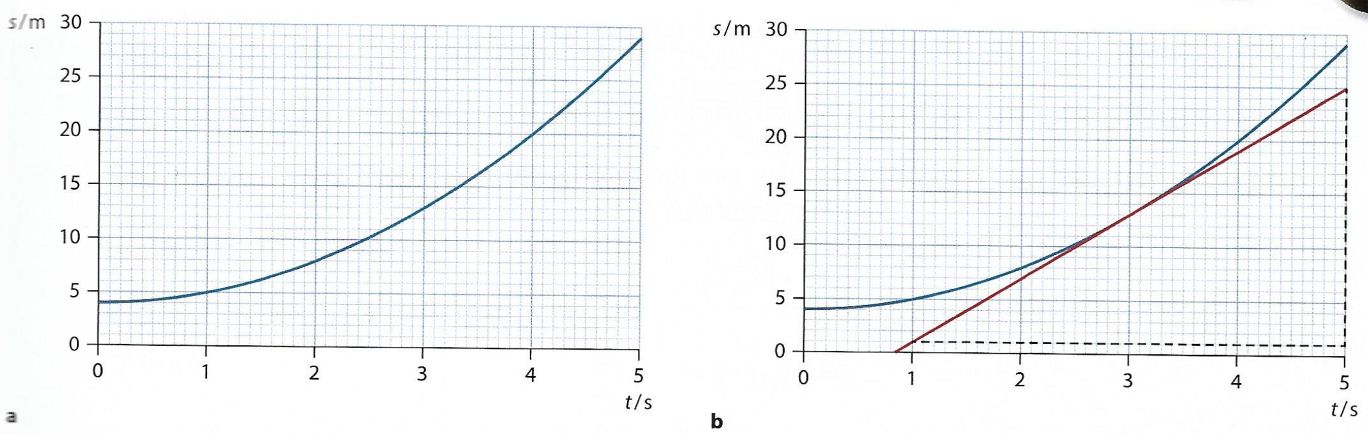


Figure 2.7 **a** In uniformly accelerated motion the graph of position versus time is a curve. **b** The slope of the tangent at a particular point gives the velocity at that point.

In Figure **2.7b** the tangent is drawn at $t = 3.0\,\mathrm{s}$. We can use this to find the instantaneous velocity at $t = 3.0\,\mathrm{s}$. The slope of this tangent line is:

$$\frac{25-1.0}{5.0-1.0} = 6.0\,\mathrm{m\,s^{-1}}$$

The slope of the tangent to the graph of position versus time is velocity

To find the instantaneous velocity at some other instant of time we must take another tangent and we will find a different instantaneous velocity. At the point at $t = 0$ it is particularly easy to find the velocity: the tangent is horizontal and so the velocity is zero.

Instantaneous velocity can be positive or negative. The magnitude of the instantaneous velocity is known as the **instantaneous speed**.

We define the **average speed** to be the total distance travelled divided by the total time taken. The **average velocity** is defined as the change in position (i.e. the displacement) divided by the time taken:

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

$$\text{average velocity} = \frac{\text{displacement}}{\text{total time taken}}$$

Consider the graph of velocity versus time in Figure **2.8**. Imagine approximating the straight line with a staircase. The area under the staircase is the change in position since at each step the velocity is constant. If we make the steps of the staircase smaller and smaller, the area under the line and the area under the staircase will be indistinguishable and so we have the general result that:

The area under the curve in a velocity versus time graph is the change in position.

From Figure **2.8** this area is (the shape is a trapezoid):

$$\Delta s = \left(\frac{u+v}{2}\right)t$$

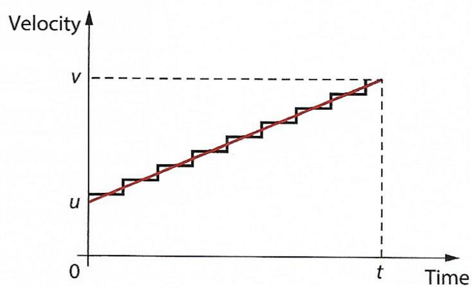


Figure 2.8 The straight-line graph may be approximated by a staircase.

But $v = u + at$, so this becomes:

$$\Delta s = \left(\frac{u + u + at}{2}\right)t = ut + \tfrac{1}{2}at^2$$

So we have two formulas for position in the case of uniformly accelerated motion (recall that $\Delta s = s - s_i$):

$$s = s_i + \left(\frac{u + v}{2}\right)t$$

$$s = s_i + ut + \tfrac{1}{2}at^2$$

We get a final formula if we combine $s = s_i + ut + \tfrac{1}{2}at^2$ with $v = u + at$. From the second equation write $t = \dfrac{v - u}{a}$ and substitute in the first equation to get:

$$s - s_i = u\frac{v - u}{a} + \frac{1}{2}\left(\frac{v + u}{a}\right)^2$$

After a bit of uninteresting algebra this becomes:

$$v^2 = u^2 + 2a(s - s_i)$$

This is useful in problems in which no information on time is given.

Graphs of position versus time for uniformly accelerated motion are parabolas (Figure **2.9**). If the parabola 'holds water' the acceleration is positive. If not, the acceleration is negative.

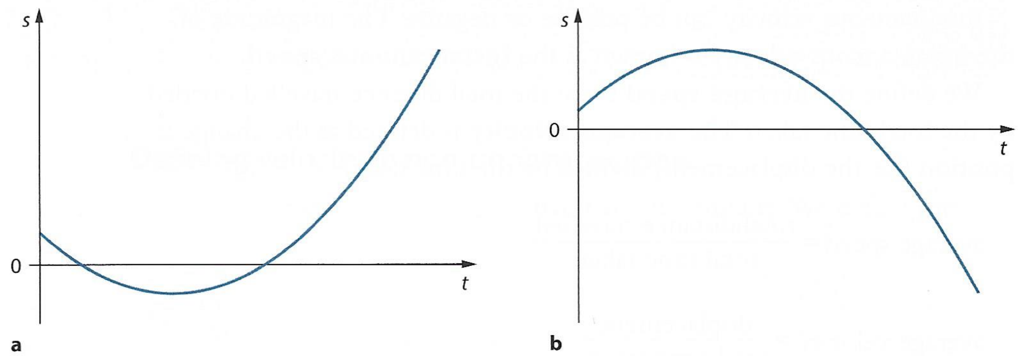


Figure 2.9 Graphs of position *s* against time *t* for uniformly accelerated motion. **a** Positive acceleration. **b** Negative acceleration.

Exam tip

The table summarises the meaning of the slope and area for the different motion graphs.

Graph of …	Slope	Area
position against time	velocity	
velocity against time	acceleration	change in position
acceleration against time		change in velocity

These formulas can be used for constant acceleration only (if the initial position is zero, Δs may be replaced by just s).

$$v = u + at \qquad \Delta s = ut + \tfrac{1}{2}at^2 \qquad \Delta s = \left(\frac{u + v}{2}\right)t \qquad v^2 = u^2 + 2a\Delta s$$

Worked examples

2.3 A particle has initial velocity $2.00\,\mathrm{m\,s^{-1}}$ and acceleration $a=4.00\,\mathrm{m\,s^{-2}}$. Find its displacement after 10.0 s.

Displacement is the change of position, i.e. $\Delta s = s - s_i$. We use the equation:

$$\Delta s = ut + \tfrac{1}{2}at^2$$

$$\Delta s = 2.00 \times 10.0 + \tfrac{1}{2} \times 4.00 \times 10.0^2$$

$$\Delta s = 220\,\mathrm{m}$$

2.4 A car has an initial velocity of $u=5.0\,\mathrm{m\,s^{-1}}$. After a displacement of 20 m, its velocity becomes $7.0\,\mathrm{m\,s^{-1}}$. Find the acceleration of the car.

Here, $\Delta s = s - s_i = 20\,\mathrm{m}$. So use $v^2 = u^2 + 2a\Delta s$ to find a.

$$7.0^2 = 5.0^2 + 2a \times 20$$

$$24 = 40a$$

Therefore $a = 0.60\,\mathrm{m\,s^{-2}}$.

2.5 A body has initial velocity $4.0\,\mathrm{m\,s^{-1}}$. After 6.0 s the velocity is $12\,\mathrm{m\,s^{-1}}$. Determine the displacement of the body in the 6.0 s.

We know u, v and t. We can use:

$$\Delta s = \left(\frac{v+u}{2}\right)t$$

to get:

$$\Delta s = \left(\frac{12+4.0}{2}\right) \times 6.0$$

$$\Delta s = 48\,\mathrm{m}$$

A slower method would be to use $v = u + at$ to find the acceleration:

$$12 = 4.0 + 6.0a$$

$$\Rightarrow \quad a = 1.333\,\mathrm{m\,s^{-2}}$$

Then use the value of a to find Δs:

$$\Delta s = ut + \tfrac{1}{2}at^2$$

$$\Delta s = 4.0 \times 6.0 + \tfrac{1}{2} \times 1.333 \times 36$$

$$\Delta s = 48\,\mathrm{m}$$

2.6 Two balls start out moving to the right with constant velocities of $5.0\,\text{m s}^{-1}$ and $4.0\,\text{m s}^{-1}$. The slow ball starts first and the other 4.0 s later. Determine the position of the balls when they meet.

Let the two balls meet t s after the first ball starts moving.

The position of the slow ball is: $s = 4t$

The position of the fast ball is: $5(t-4)$

(The factor $t-4$ is there because after t s the fast ball has actually been moving for only $t-4$ seconds.)

These two positions are equal when the two balls meet, and so:

$4t = 5t - 20$

$\Rightarrow \quad t = 20\,\text{s}$

Substituting into the equation for the position of the slow ball, the position where the balls meet is 80 m to the right of the start.

2.7 A particle starts out from the origin with velocity $10\,\text{m s}^{-1}$ and continues moving at this velocity for 5 s. The velocity is then abruptly reversed to $-5\,\text{m s}^{-1}$ and the object moves at this velocity for 10 s. For this motion find:

a the change in position, i.e. the displacement
b the total distance travelled
c the average speed
d the average velocity.

The problem is best solved using the velocity–time graph, which is shown in Figure **2.10**.

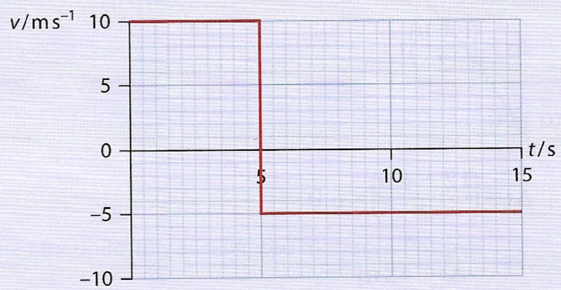


Figure 2.10

a The initial position is zero. Thus, after 5.0 s the position is $10 \times 5.0\,\text{m} = 50\,\text{m}$ (the area under the first part of the graph). In the next 10 s the displacement changes by $-5.0 \times 10 = -50\,\text{m}$ (the area under the second part of the graph). The change in position, i.e. the displacement, is thus $50 - 50 = 0\,\text{m}$.

b Take the initial velocity as moving to the right. The object moved toward the right, stopped and returned to its starting position (we know this because the displacement was 0). The distance travelled is 50 m in moving to the right and 50 m coming back, giving a total distance travelled of 100 m.

c The average speed is $\dfrac{100\,\text{m}}{15\,\text{s}} = 6.7\,\text{m s}^{-1}$.

d The average velocity is zero, since the displacement is zero.

2.8 An object with initial velocity $20\,\mathrm{m\,s^{-1}}$ and initial position of $-75\,\mathrm{m}$ experiences a constant acceleration of $-2\,\mathrm{m\,s^{-2}}$. Sketch the position–time graph for this motion for the first 20 s.

Use the equation $s = ut + \frac{1}{2}at^2$. Substituting the values we know, the displacement is given by $s = -75 + 20t - t^2$. This is the function we must graph. The result is shown in Figure **2.11.**

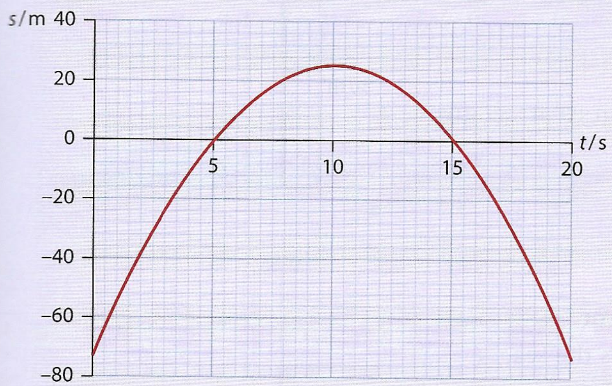


Figure 2.11

At 5 s the object reaches the origin and overshoots it. It returns to the origin 10 s later ($t = 15\,\mathrm{s}$). The furthest it gets from the origin is 25 m. The velocity at 5 s is $10\,\mathrm{m\,s^{-1}}$ and at 15 s it is $-10\,\mathrm{m\,s^{-1}}$. At 10 s the velocity is zero.

A special acceleration

Assuming that we can neglect air resistance and other frictional forces, an object thrown into the air will experience the **acceleration of free fall** while in the air. This is an acceleration caused by the attraction between the Earth and the body. The magnitude of this acceleration is denoted by g. Near the surface of the Earth $g = 9.8\,\mathrm{m\,s^{-2}}$. The direction of this acceleration is always vertically downward. (We will sometimes approximate g by $10\,\mathrm{m\,s^{-2}}$.)

Worked example

2.9 An object is thrown vertically upwards with an initial velocity of $20\,\mathrm{m\,s^{-1}}$ from the edge of a cliff that is 30 m from the sea below, as shown in Figure **2.12.**

Determine:

a the ball's maximum height

b the time taken for the ball to reach its maximum height

c the time to hit the sea

d the speed with which it hits the sea.

(You may approximate g by $10\,\mathrm{m\,s^{-2}}$.)

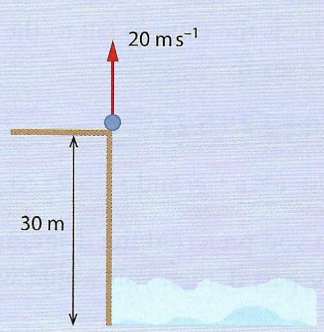


Figure 2.12 A ball is thrown upwards from the edge of a cliff.

We have motion on a vertical line so we will use the symbol y for position (Figure **2.13a**). We make the vertical line point upwards. The zero for displacement is the ball's initial position.

a The quickest way to get the answer to this part is to use $v^2 = u^2 - 2gy$. (The acceleration is $a = -g$.) At the highest point $v = 0$, and so:

$$0 = 20^2 - 2 \times 10y$$

$$\Rightarrow y = 20\,\text{m}$$

b At the highest point the object's velocity is zero. Using $v = 0$ in $v = u - gt$ gives:

$$0 = 20 - 10 \times t$$

$$t = \frac{20}{10} = 2.0\,\text{s}$$

c There are many ways to do this. One is to use the displacement arrow shown in blue in Figure **2.13a**. Then when the ball hits the sea, $y = -30\,\text{m}$. Now use the formula $y = ut - \frac{1}{2}gt^2$ to find an equation that only has the variable t:

$$-30 = 20 \times t - 5 \times t^2$$

$$t^2 - 4t - 6 = 0$$

This is a quadratic equation. Using your calculator you can find the two roots as $-1.2\,\text{s}$ and $5.2\,\text{s}$. Choose the positive root to find the answer $t = 5.2\,\text{s}$.

Another way of looking at this is shown in Figure **2.13b**. Here we start at the highest point and make the line along which the ball moves point downwards. Then, at the top $y = 0$, at the sea $y = +50$ and $g = +10\,\text{m}\,\text{s}^{-2}$. Now, the initial velocity is zero because we take our initial point to be at the top.

Using $y = ut + \frac{1}{2}gt^2$ with $u = 0$, we find:

$$50 = 5t^2$$

$$\Rightarrow t = 3.2\,\text{s}$$

This is the time to fall to the sea. It took 2.0 s to reach the highest point, so the total time from launch to hitting the sea is:

$$2.0 + 3.2 = 5.2\,\text{s}.$$

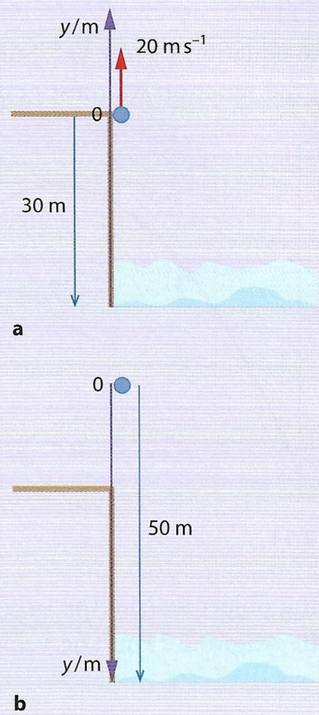


Figure 2.13 Diagrams for solving the ball's motion. **a** Displacement upwards is positive. **b** The highest point is the zero of displacement.

d Use $v = u - gt$ and $t = 5.2\,\text{s}$ to get $v = 20 - 10 \times 5.2 = -32\,\text{m}\,\text{s}^{-1}$. The speed is then $32\,\text{m}\,\text{s}^{-1}$.

(If you preferred the diagram in Figure **2.13b** for working out part **c** and you want to continue this method for part **d**, then you would write $v = u + gt$ with $t = 3.2\,\text{s}$ and $u = 0$ to get $v = 10 \times 3.2 = +32\,\text{m}\,\text{s}^{-1}$.)

Projectile motion

Figure **2.14** shows the positions of two objects every 0.2 s: the first was simply allowed to drop vertically from rest, the other was launched horizontally with no vertical component of velocity. We see that in the vertical direction, both objects fall the **same distance** in the **same time**.

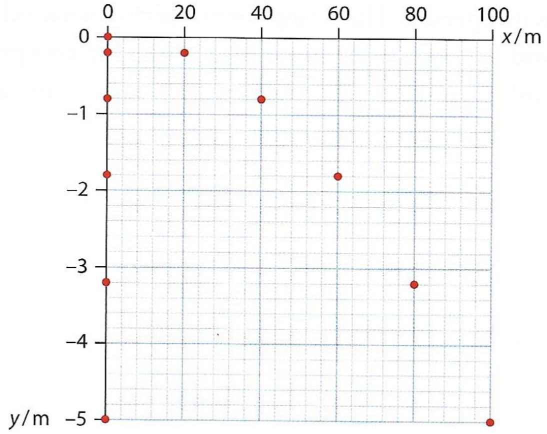


Figure 2.14 A body dropped from rest and one launched horizontally cover the same vertical displacement in the same time.

How do we understand this fact? Consider Figure **2.15**, in which a black ball is projected horizontally with velocity v. A blue ball is allowed to drop vertically from the same height. Figure **2.15a** shows the situation when the balls are released as seen by an observer X at rest on the ground. But suppose there is an observer Y, who moves to the right with velocity $\frac{v}{2}$ with respect to the ground. What does Y see? Observer Y sees the black ball moving to the right with velocity $\frac{v}{2}$ and the blue ball approaching with velocity $-\frac{v}{2}$ (Figure **2.15b**) The motions of the two balls are therefore **identical** (except for direction). So this observer will determine that the two bodies reach the ground at the **same time**. Since time is absolute in Newtonian physics, the two bodies must reach the ground at the same time as far as any other observer is concerned as well.

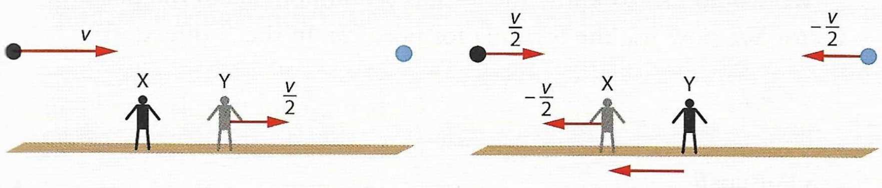


Figure 2.15 **a** A ball projected horizontally and one simply dropped from rest from the point of view of observer X. Observer Y is moving to the right with velocity $\frac{v}{2}$ with respect to the ground. **b** From the point of view of observer Y, the black and the blue balls have identical motions.

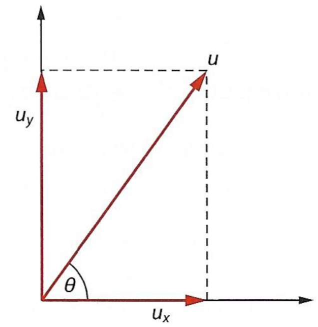


Figure 2.16 A projectile is launched at an angle θ to the horizontal with speed u.

The discussion shows that the motion of a ball that is projected at some angle can be analysed by separately looking at the horizontal and the vertical directions. All we have to do is consider two motions, one in the horizontal direction in which there is no acceleration, and another in the vertical direction in which we have an acceleration, g.

Consider Figure **2.16**, where a projectile is launched at an angle θ to the horizontal with speed u. The components of the *initial* velocity vector are $u_x = u\cos\theta$ and $u_y = u\sin\theta$. At some later time t the components of velocity are v_x and v_y. In the x-direction we do not have any acceleration and so:

$$v_x = u_x$$

$$v_x = u\cos\theta$$

In the y-direction the acceleration is $-g$ and so:

$$v_y = u_y - gt$$

$$v_y = u\sin\theta - gt$$

The green vector in Figure **2.17a** shows the position of the projectile t seconds after launch. The red arrows in Figure **2.17b** show the velocity vectors.

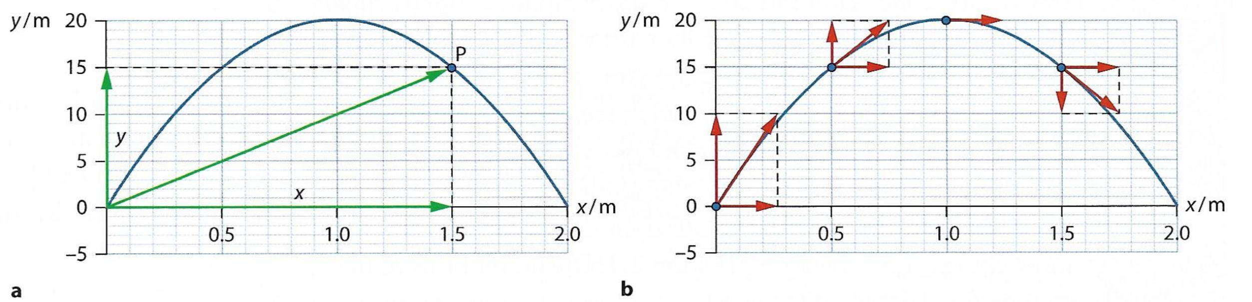


Figure 2.17 **a** The position of the particle is determined if we know the x- and y-components of the position vector. **b** The velocity vectors for projectile motion are tangents to the parabolic path.

Exam tip

All that we are doing is using the formulas from the previous section for velocity and position $v = u + at$ and $s = ut + \frac{1}{2}at^2$ and rewriting them **separately** for each direction x and y.

In the x-direction there is zero acceleration and in the y-direction there is an acceleration $-g$.

We would like to know the x- and y-components of the position vector. We now use the formula for position. In the x-direction:

$$x = u_x t$$

$$x = ut\cos\theta$$

And in the y-direction:

$$y = u_y t - \tfrac{1}{2}gt^2$$

$$y = ut\sin\theta - \tfrac{1}{2}gt^2$$

Let us collect what we have derived so far. We have four equations with which we can solve any problem with projectiles, as we will soon see:

$$\underbrace{v_x = u\cos\theta}_{x\text{-velocity}}, \qquad \underbrace{v_y = u\cos\theta - gt}_{y\text{-velocity}}$$

$$\underbrace{x = ut\cos\theta}_{x\text{-displacement}}, \qquad \underbrace{y = ut\sin\theta - \frac{1}{2}gt^2}_{y\text{-displacement}}$$

The equation with 'squares of speeds' is a bit trickier (carefully review the following steps). It is:

$$v^2 = u^2 - 2gy$$

Since $v^2 = v_x^2 + v_y^2$ and $u^2 = u_x^2 + u_y^2$, and in addition $v_x^2 = u_x^2$, this is also equivalent to:

$$v_y^2 = u_y^2 - 2gy$$

Exam tip
Always choose your *x*- and *y*-axes so that the origin is the point where the launch takes place.

Worked examples

2.10 A body is launched with a speed of $18.0\,\text{m s}^{-1}$ at the following angles:

a 30° to the horizontal
b 0° to the horizontal
c 90° to the horizontal.

Find the *x*- and *y*-components of the initial velocity in each case.

a $v_x = u\cos\theta$ $\qquad$ $v_y = u\sin\theta$

$v_x = 18.0 \times \cos 30°$ $\qquad$ $v_y = 18.0 \times \sin 30°$

$v_x = 15.6\,\text{m s}^{-1}$ $\qquad$ $v_y = 9.00\,\text{m s}^{-1}$

b $v_x = 18.0\,\text{m s}^{-1}$ $\qquad$ $v_y = 0\,\text{m s}^{-1}$

c $v_x = 0$ $\qquad$ $v_y = 18.0\,\text{m s}^{-1}$

2.11 Sketch graphs to show the variation with time of the horizontal and vertical components of velocity for a projectile launched at some angle above the horizontal.

The graphs are shown in Figure **2.18**.

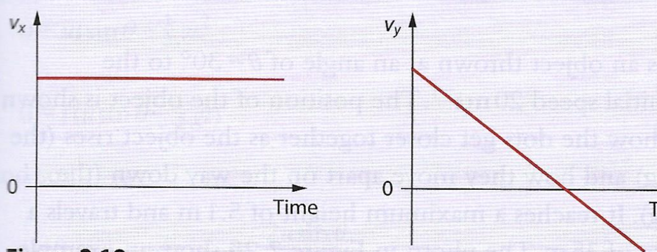


Figure 2.18

2.12 An object is launched horizontally from a height of 20 m above the ground with speed 15 m s^{-1}. Determine:

a the time at which it will hit the ground

b the horizontal distance travelled

c the speed with which it hits the ground.

(Take $g = 10\,\text{m s}^{-2}$.)

a The launch is horizontal, i.e. $\theta = 0°$, and so the formula for vertical displacement is just $y = -\frac{1}{2}gt^2$.

The object will hit the ground when $y = -20$ m.

Substituting the values, we find:

$$-20 = -5t^2$$

$$\Rightarrow \quad t = 2.0\,\text{s}$$

Exam tip

This is a basic problem – you must know how to do this!

b The horizontal distance is found from $x = ut$. Substituting values:

$$x = 15 \times 2.0 = 30\,\text{m}$$

(Remember that $\theta = 0°$).

c Use $v^2 = u^2 - 2gy$ to get:

$$v^2 = 15^2 - 2 \times 10 \times (-20)$$

$$v = 25\,\text{m s}^{-1}$$

2.13 An object is launched horizontally with a velocity of 12 m s^{-1}. Determine:

a the vertical component of velocity after 4.0 s

b the x- and y-components of the position vector of the object after 4.0 s.

a The launch is again horizontal, i.e. $\theta = 0°$, so substitute this value in the formulas. The horizontal component of velocity is 12 m s^{-1} at all times.

From $v_y = -gt$, the vertical component after 4.0 s is $v_y = -20\,\text{m s}^{-1}$.

b The coordinates after time t are:

$x = ut$ and $y = -\frac{1}{2}gt^2$

$x = 12.0 \times 4.0$ $\quad y = -5 \times 16$

$x = 48\,\text{m}$ $\quad y = -80\,\text{m}$

Figure **2.19** shows an object thrown at an angle of $\theta = 30°$ to the horizontal with initial speed 20 m s^{-1}. The position of the object is shown every 0.2 s. Note how the dots get closer together as the object rises (the speed is decreasing) and how they move apart on the way down (the speed is increasing). It reaches a maximum height of 5.1 m and travels a horizontal distance of 35 m. The photo in Figure **2.20** show an example of projectile motion.

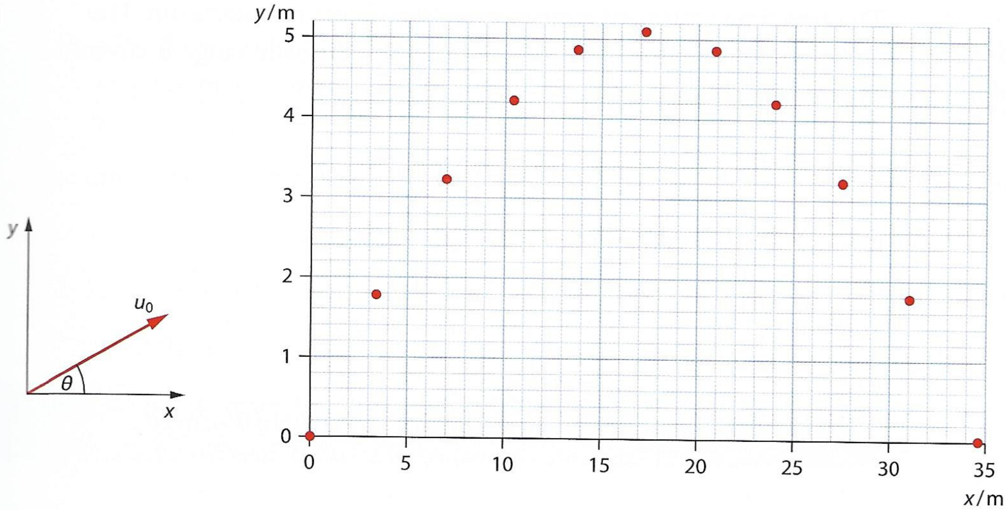


Figure 2.19 A launch at of $\theta = 30°$ to the horizontal with initial speed $20\,\mathrm{m\,s^{-1}}$.

At what point in time does the vertical velocity component become zero? Setting $v_y = 0$ we find:

$$0 = u\sin\theta - gt$$

$$\Rightarrow \quad t = \frac{u\sin\theta}{g}$$

The time when the vertical velocity becomes zero is, of course, the time when the object attains its maximum height. What is this height? Going back to the equation for the vertical component of displacement, we find that when:

$$t = \frac{u\sin\theta}{g}$$

y is given by:

$$y_{\max} = u\frac{u\sin\theta}{g}\sin\theta - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$y_{\max} = \frac{u^2\sin^2\theta}{2g}$$

What about the maximum displacement in the horizontal direction (sometimes called the range)? At this point the vertical component of displacement y is zero. Setting $y = 0$ in the formula for y gives:

$$0 = ut\sin\theta - \tfrac{1}{2}gt^2$$

$$0 = t\left(u\sin\theta - \tfrac{1}{2}gt\right)$$

and so:

$$t = 0 \qquad \text{and} \qquad t = \frac{2u\sin\theta}{g}$$

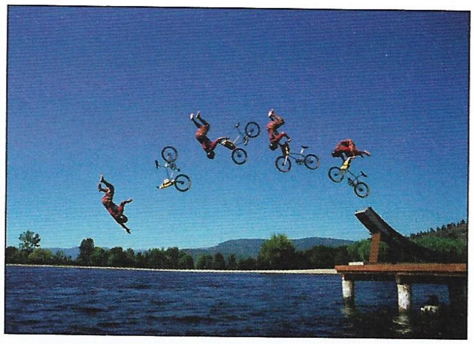


Figure 2.20 A real example of projectile motion!

Exam tip
You should not remember these formulas by heart. You should be able to derive them quickly.

The first time $t = 0$ is, of course, when the object first starts out. The second time is what we want – the time in which the range is covered. Therefore the range is:

$$x = \frac{2u^2 \sin\theta \cos\theta}{g}$$

A bit of trigonometry allows us to rewrite this as:

$$x = \frac{u^2 \sin(2\theta)}{g}$$

One of the identities in trigonometry is $2\sin\theta\cos\theta = \sin 2\theta$

The maximum value of $\sin 2\theta$ is 1, and this happens when $2\theta = 90°$ (i.e. $\theta = 45°$); in other words, we obtain the maximum range with a launch angle of 45°. This equation also says that there are two different angles of launch that give the same range for the same initial speed. These two angles add up to a right angle (can you see why?).

Worked examples

2.14 A projectile is launched at 32.0° to the horizontal with initial speed $25.0\,\text{m s}^{-1}$. Determine the maximum height reached. (Take $g = 9.81\,\text{m s}^{-2}$.)

The vertical velocity is given by $v_y = u\sin\theta - gt$ and becomes zero at the highest point. Thus:

$$t = \frac{u\sin\theta}{g}$$

$$t = \frac{25.0 \times \sin 32.0°}{9.81}$$

$$t = 1.35\,\text{s}$$

Substituting in the formula for y, $y = ut\sin\theta - \frac{1}{2}gt^2$, we get:

$$y = 25 \times \sin 32.0° \times 1.35 - \tfrac{1}{2} \times 9.81 \times 1.35^2$$

$$y = 8.95\,\text{m}$$

2.15 A projectile is launched horizontally from a height of 42 m above the ground. As it hits the ground, the velocity makes an angle of 55° to the horizontal. Find the initial velocity of launch. (Take $g = 9.8\,\mathrm{m\,s^{-2}}$.)

The time it takes to hit the ground is found from $y = \frac{1}{2}gt^2$ (here $\theta = 0°$ since the launch is horizontal).

The ground is at $y = -42\,\mathrm{m}$ and so:

$$-42 = -\frac{1}{2} \times 9.8t^2$$

$$\Rightarrow \quad t = 2.928\,\mathrm{s}$$

Using $v = u - at$, when the projectile hits the ground:

$$v_y = 0 - 9.8 \times 2.928$$

$$v_y = -28.69\,\mathrm{m\,s^{-1}}$$

We know the angle the final velocity makes with the ground (Figure **2.21**). Hence:

$$\tan 55° = \left|\frac{v_y}{v_x}\right|$$

$$\Rightarrow \quad v_x = \frac{28.69}{\tan 55°}$$

$$v_x = 20.03 \approx 20\,\mathrm{m\,s^{-1}}$$

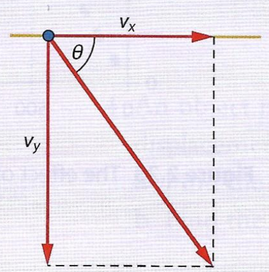


Figure 2.21

Fluid resistance

The discussion of the previous sections has neglected air resistance forces. In general, whenever a body moves through a fluid (gas or liquid) it experiences a **fluid resistance force** that is directed opposite to the velocity. Typically $F = kv$ for low speeds and $F = kv^2$ for high speeds (where k is a constant). The magnitude of this force increases with increasing speed.

Imagine dropping a body of mass m from some height. Assume that the force of air resistance on this body is $F = kv$. Initially, the only force on the body is its weight, which accelerates it downward. As the speed increases, the force of air resistance also increases. Eventually, this force will become equal to the weight and so the acceleration will become zero: the body will then move at constant speed, called **terminal speed**, v_T. This speed can be found from:

$$mg = kv_T$$

which leads to:

$$v_T = \frac{mg}{k}$$

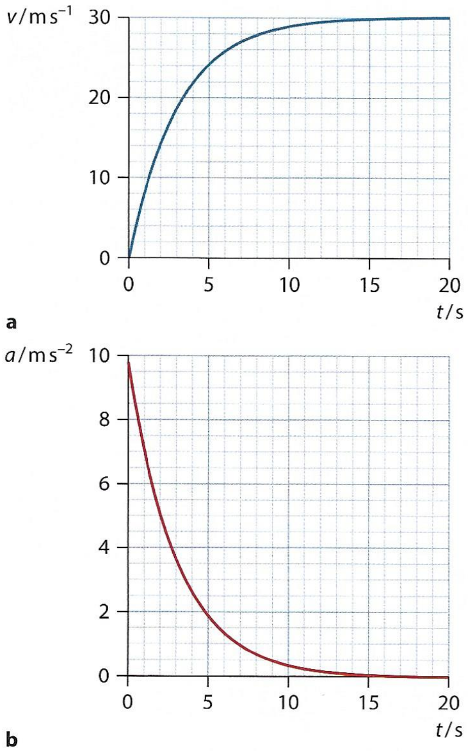


Figure 2.22 The variation with time of **a** speed and **b** acceleration in motion with an air resistance force proportional to speed.

Figure **2.22** shows how the speed and acceleration vary for motion with an air resistance force that is proportional to speed. The speed eventually becomes the terminal speed and the acceleration becomes zero. The initial acceleration is g.

The effect of air resistance forces on projectiles is very pronounced. Figure **2.23** shows the positions of a projectile with (red) and without (blue) air resistance forces. With air resistance forces the range and maximum height are smaller and the shape is no longer symmetrical. The projectile hits the ground with a steeper angle.

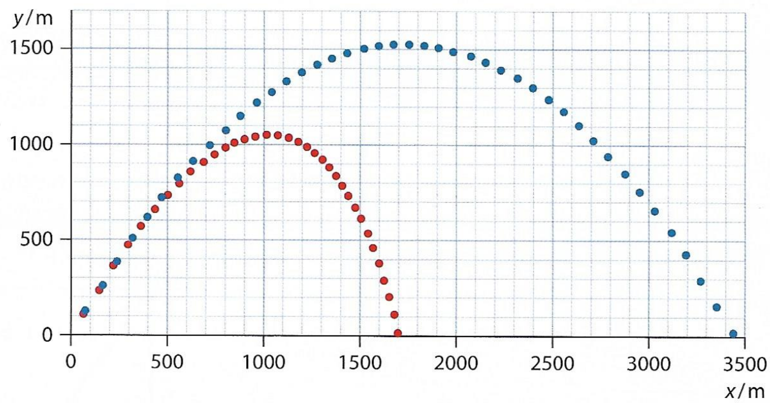


Figure 2.23 The effect of air resistance on projectile motion.

Worked example

2.16 The force of air resistance in the motion described by Figure **2.22** is given by $F = 0.653v$. Determine the mass of the projectile.

The particle is getting slower. At some point it will stop instantaneously, i.e. its velocity v will be zero.

We know that $v = u + at$. Just substituting values gives:

$$0 = 12 + (-3.0) \times t$$

$$3.0t = 12$$

Hence $t = 4.0\,\text{s}$.

The terminal speed is $30\,\text{m s}^{-1}$ and is given by $v_T = \frac{mg}{k}$. Hence:

$$m = \frac{kv_T}{g}$$

$$m = \frac{0.653 \times 30}{9.8}$$

$$m \approx 2.0\,\text{kg}$$

Nature of science

The simple and the complex

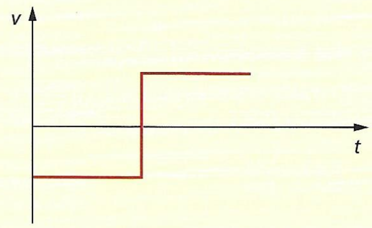
Careful observation of motion in the natural world led to the equations for motion with uniform acceleration along a straight line that we have used in this section. Thinking about what causes an object to move links to the idea of forces. However, although the material in this section is perhaps some of the 'easiest' material in your physics course, it does not enable one to understand the falling of a leaf off a tree. The falling leaf is complicated because it is acted upon by several forces: its weight, but also by air resistance forces that constantly vary as the orientation and speed of the leaf change. In addition, there is wind to consider as well as the fact that turbulence in air greatly affects the motion of the leaf. So the physics of the falling leaf is far away from the physics of motion along a straight line at constant acceleration. But learning the principles of physics in a simpler context allows its application in more involved situations.

? Test yourself

Uniform motion

1 A car must be driven a distance of 120 km in 2.5 h. During the first 1.5 h the average speed was $70\,\mathrm{km\,h^{-1}}$. Calculate the average speed for the remainder of the journey.

2 Draw the position–time graph for an object moving in a straight line with a velocity–time graph as shown below. The initial position is zero. You do not have to put any numbers on the axes.



3 Two cyclists, **A** and **B**, have displacements 0 km and 70 km, respectively. At $t = 0$ they begin to cycle towards each other with velocities $15\,\mathrm{km\,h^{-1}}$ and $20\,\mathrm{km\,h^{-1}}$, respectively. At the same time, a fly that was sitting on **A** starts flying towards **B** with a velocity of $30\,\mathrm{km\,h^{-1}}$. As soon as the fly reaches **B** it immediately turns around and flies towards **A**, and so on until **A** and **B** meet.

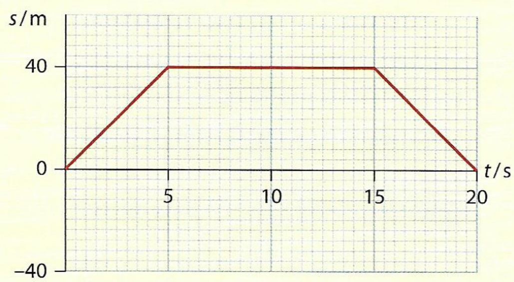
a Find the position of the two cyclists and the fly when all three meet.

b Determine the distance travelled by the fly.

4 An object moving in a straight line has the displacement–time graph shown.

a Find the average speed for the trip.

b Find the average velocity for the trip.



Accelerated motion

5 The initial velocity of a car moving on a straight road is $2.0\,\mathrm{m\,s^{-1}}$. It becomes $8.0\,\mathrm{m\,s^{-1}}$ after travelling for 2.0 s under constant acceleration. Find the acceleration.

6 A car accelerates from rest to $28\,\mathrm{m\,s^{-1}}$ in 9.0 s. Find the distance it travels.

7 A particle has an initial velocity of $12\,\mathrm{m\,s^{-1}}$ and is brought to rest over a distance of 45 m. Find the acceleration of the particle.

8 A particle at the origin has an initial velocity of $-6.0\,\mathrm{m\,s^{-1}}$ and moves with an acceleration of $2.0\,\mathrm{m\,s^{-2}}$. Determine when its position will become 16 m.

9 A plane starting from rest takes 15.0 s to take off after speeding over a distance of 450 m on the runway with constant acceleration. Find the take-off velocity.

10 A car is travelling at $40.0\,\mathrm{m\,s^{-1}}$. The driver sees an emergency ahead and 0.50 s later slams on the brakes. The deceleration of the car is $4.0\,\mathrm{m\,s^{-2}}$.

a Find the distance travelled before the car stops.

b Calculate the stopping distance if the driver could apply the brakes instantaneously without a reaction time.

c Calculate the difference in your answers to **a** and **b**.

d Assume now that the car was travelling at $30.0\,\mathrm{m\,s^{-1}}$ instead. Without performing any calculations, state whether the answer to **c** would now be less than, equal to or larger than before. Explain your answer.

11 Two balls are dropped from rest from the same height. One of the balls is dropped 1.00 s after the other.

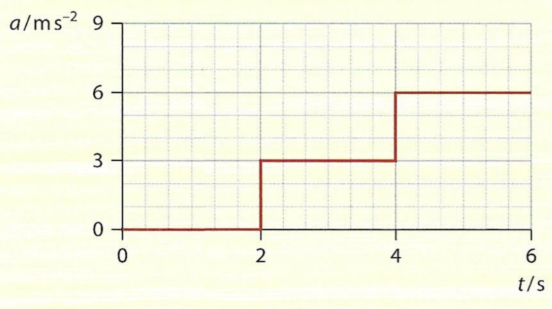
a Find the distance that separates the two balls 2.00 s after the second ball is dropped.

b State what happens to the distance separating the balls as time goes on.

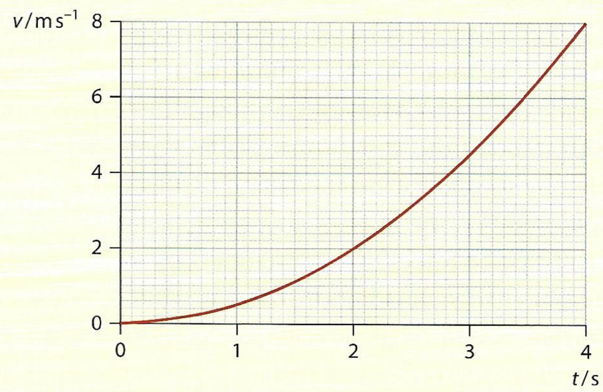
12 A particle moves in a straight line with an acceleration that varies with time as shown in the diagram. Initially the velocity of the object is $2.00\,\mathrm{m\,s^{-1}}$.

a Find the maximum velocity reached in the first 6.00 s of this motion.

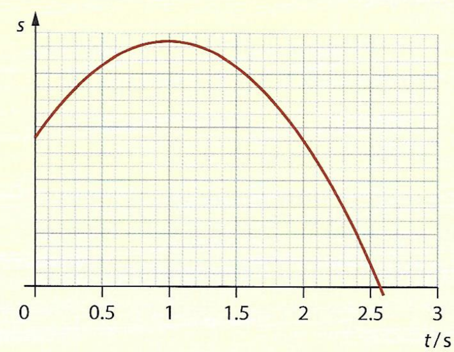
b Draw a graph of the velocity versus time.



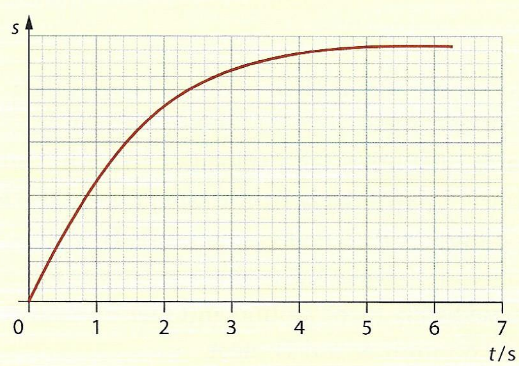
13 The graph shows the variation of velocity with time of an object. Find the acceleration at 2.0 s.



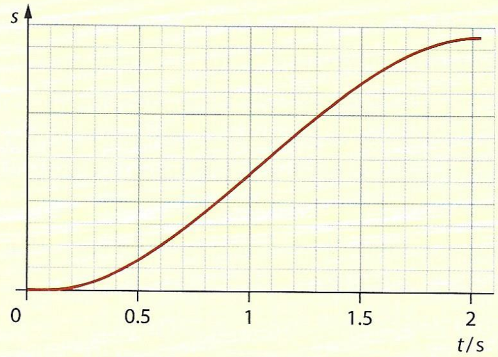
14 The graph shows the variation of the position of a moving object with time. Draw the graph showing the variation of the velocity of the object with time.



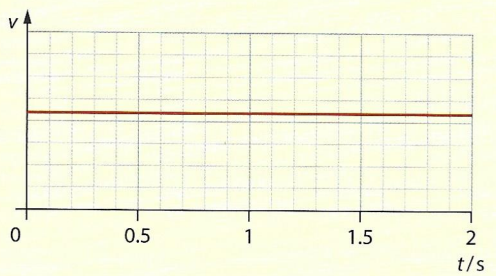
15 The graph shows the variation of the position of a moving object with time. Draw the graph showing the variation of the velocity of the object with time.



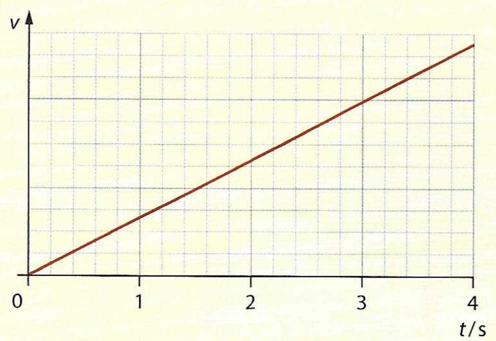
16 The graph shows the variation of the position of a moving object with time. Draw the graph showing the variation of the velocity of the object with time.



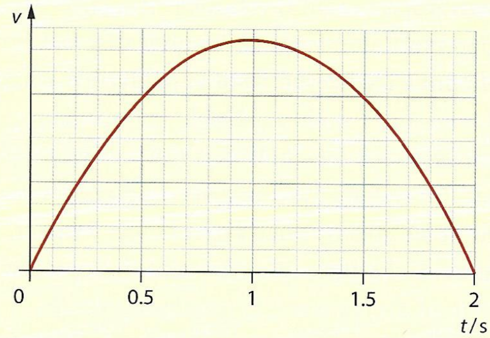
17 The graph shows the variation of the velocity of a moving object with time. Draw the graph showing the variation of the position of the object with time.



18 The graph shows the variation of the velocity of a moving object with time. Draw the graph showing the variation of the position of the object with time (assuming a zero initial position).

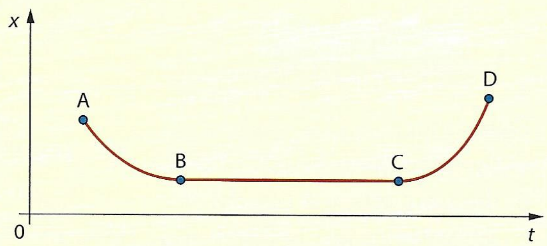


19 The graph shows the variation of the velocity of a moving object with time. Draw the graph showing the variation of the acceleration of the object with time.



20 Your brand new convertible Ferrari is parked 15 m from its garage when it begins to rain. You do not have time to get the keys, so you begin to push the car towards the garage. The maximum acceleration you can give the car is $2.0\,\mathrm{m\,s^{-2}}$ by pushing and $3.0\,\mathrm{m\,s^{-2}}$ by pulling back on the car. Find the least time it takes to put the car in the garage. (Assume that the car, as well as the garage, are point objects.)

21 The graph shows the displacement versus time of an object moving in a straight line. Four points on this graph have been selected.



a Is the velocity between **A** and **B** positive, zero or negative?

b What can you say about the velocity between **B** and **C**?

c Is the acceleration between **A** and **B** positive, zero or negative?

d Is the acceleration between **C** and **D** positive, zero or negative?

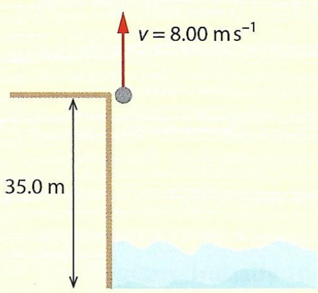
22 Sketch velocity–time sketches (no numbers are necessary on the axes) for the following motions.

a A ball is dropped from a certain height and bounces off a hard floor. The speed just before each impact with the floor is the same as the speed just after impact. Assume that the time of contact with the floor is negligibly small.

b A cart slides with negligible friction along a horizontal air track. When the cart hits the ends of the air track it reverses direction with the same speed it had right before impact. Assume the time of contact of the cart and the ends of the air track is negligibly small.

c A person jumps from a hovering helicopter. After a few seconds she opens a parachute. Eventually she will reach a terminal speed and will then land.

23 A stone is thrown vertically up from the edge of a cliff 35.0 m from the sea. The initial velocity of the stone is $8.00\,\mathrm{m\,s^{-1}}$.



Determine:

a the maximum height of the stone

b the time when it hits the sea

c the velocity just before hitting the sea

d the distance the stone covers

e the average speed and the average velocity for this motion.

24 A ball is thrown upward from the edge of a cliff with velocity $20.0\,\mathrm{m\,s^{-1}}$. It reaches the bottom of the cliff 6.0 s later.

a Determine the height of the cliff.

b Calculate the speed of the ball as it hits the ground.

Projectile motion

25 A ball rolls off a table with a horizontal speed of $2.0\,\mathrm{m\,s^{-1}}$. The table is 1.3 m high. Calculate how far from the table the ball will land.

26 Two particles are on the same vertical line. They are thrown horizontally with the same speed, $4.0\,\mathrm{m\,s^{-1}}$, from heights of 4.0 m and 8.0 m.

a Calculate the distance that will separate the two objects when both land on the ground.

b The particle at the 4.0 m height is now launched with horizontal speed u such that it lands at the same place as the particle launched from 8.0 m. Calculate u.

27 For an object thrown at an angle of 40° to the horizontal at a speed of $20\,\mathrm{m\,s^{-1}}$, draw graphs of:

a horizontal velocity against time

b vertical velocity against time

c acceleration against time.

28 Determine the maximum height reached by an object thrown with speed $24\,\mathrm{m\,s^{-1}}$ at 40° to the horizontal.

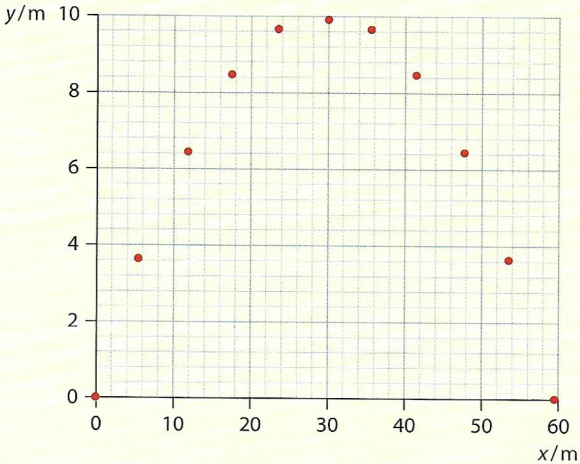
29 An object is thrown with speed $20.0\,\mathrm{m\,s^{-1}}$ at an angle of 50° to the horizontal. Draw graphs to show the variation with time of:

a the horizontal position

b the vertical position.

30 A cruel hunter takes aim horizontally at a chimp that is hanging from the branch of a tree, as shown in the diagram. The chimp lets go of the branch as soon as the hunter pulls the trigger. Treating the chimp and the bullet as point particles, determine if the bullet will hit the chimp.

31 A ball is launched from the surface of a planet. Air resistance and other frictional forces are neglected. The graph shows the position of the ball every 0.20 s.



a Use this graph to determine:

i the components of the initial velocity of the ball

ii the angle to the horizontal the ball was launched at

iii the acceleration of free fall on this planet.

b Make a copy of the graph and draw two arrows to represent the velocity and the acceleration vectors of the ball at $t = 1.0$ s.

c The ball is now launched under identical conditions from the surface of a **different** planet where the acceleration due to gravity is twice as large. Draw the path of the ball on your graph.

32 A stone is thrown with a speed of $20.0\,\mathrm{m\,s^{-1}}$ at an angle of 48° to the horizontal from the edge of a cliff 60.0 m above the surface of the sea.

a Calculate the velocity with which the stone hits the sea.

b Discuss qualitatively the effect of air resistance on your answer to **a**.

33 **a** State what is meant by **terminal speed**.

b A ball is dropped from rest. The force of air resistance in the ball is proportional to the ball's speed. Explain why the ball will reach terminal speed.