Learning objectives

- Be able to re-formulate Newton's second law when the mass is variable.
- Understand the concept of impulse and be able to analyse force–time graphs.
- Be able to derive and apply the law of conservation of momentum.
- Analyse elastic and inelastic collisions and explosions.

2.4 Momentum and impulse

This section introduces the concept of linear momentum, which is a very useful and powerful concept in physics. Newton's second law is expressed in terms of momentum. The law of conservation of linear momentum makes it possible to predict the outcomes in very many physical situations.

Newton's second law in terms of momentum

We saw earlier that Newton's second law was expressed as $F_{net} = ma$. In fact, this equation is only valid when the mass of the system remains constant. But there are plenty of situations where the mass does *not* remain constant. In cases where the mass changes, a different version of the second law must be used. Examples include:

- the motion of a rocket, where the mass decreases due to burnt fuel ejected away from the rocket
- sand falling on a conveyor belt so the mass increases
- a droplet of water falling through mist and increasing in mass as more water condenses.

We define a new concept, **linear momentum**, p, to be the product of the mass of a body times its velocity:

$$p = mv$$

Momentum is a vector and has the direction of the velocity. Its unit is $\mathrm{kg\,m\,s^{-1}}$ or the equivalent N s.

In terms of momentum, Newton's second law is:

$$F_{net} = \frac{\Delta p}{\Delta t}$$

The average net force on a system is equal to the rate of change of the momentum of the system.

It is easy to see that if the mass stays constant, then this version reduces to the usual ma:

$$F_{net} = \frac{\Delta p}{\Delta t} = \frac{p_{final} - p_{initial}}{\Delta t}$$

$$= \frac{mv_{final} - mv_{initial}}{\Delta t}$$

$$= m\left(\frac{v_{final} - v_{initial}}{\Delta t}\right)$$

$$= \frac{m\Delta v}{\Delta t}$$

$$F_{net} = ma$$

Worked examples

2.45 A cart moves in a horizontal line with constant speed v. Rain starts to fall and the cart fills with water at a rate of $\sigma\,\mathrm{kg\,s^{-1}}$. (This means that in one second, σ kg have fallen on the cart.) The cart must keep moving at constant speed. Determine the force that must be applied on the cart.

Exam tip
Worked example **2.45** should alert you right away that you must be careful when mass changes. Zero acceleration does not imply zero net force in this case.

Notice right away that if $F_{net} = ma$ (we drop the bold italic of the vector notation) were valid, the force would have to be zero since the car is not accelerating. But we do need a force to act on the cart because the momentum of the cart is increasing (because the mass is increasing). This force is:

$$F_{net} = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{v\Delta m}{\Delta t} = v\sigma$$

Putting some real values in, if $\sigma = 0.20\,\mathrm{kg\,s^{-1}}$ and $v = 3.5\,\mathrm{m\,s^{-1}}$, the force would have to be 0.70 N.

2.46 Gravel falls vertically on a conveyor belt at a rate of $\sigma\,\mathrm{kg\,s^{-1}}$, as shown in Figure **2.76**.

This very popular exam question is similar to Worked example **2.45**, but is worth doing again.

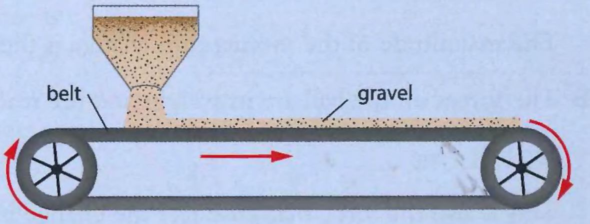


Figure 2.76

a Determine:

i the force that must be applied on the belt to keep it moving at constant speed v

ii the power that must be supplied by the motor turning the belt

iii the rate at which the kinetic energy of the gravel is changing.

b Explain why the answers to **a ii** and **a iii** are different.

a **i** The force is:

$$F_{net} = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{v\Delta m}{\Delta t} = v\sigma$$

ii The power is found from $P = Fv$. Substituting for F:

$$P = (v\sigma)v = \sigma v^2$$

iii In 1 second the mass on the belt increases by σ kg. The kinetic energy of this mass is:

$$E_K = \tfrac{1}{2}\sigma v^2$$

This is the increase in kinetic energy in a time of 1 s, so the rate of kinetic energy increase is $\frac{1}{2}\sigma v^2$.

b The rate of increase in kinetic energy is less than the power supplied. This is because the power supplied by the motor goes to increase the kinetic energy of the gravel and also to provide the energy needed to accelerate the gravel from 0 to speed v in the short interval of time when the gravel slides on the belt before achieving the constant final speed v.

2.47 A 0.50 kg ball is dropped from rest above a hard floor. When it reaches the floor it has a velocity of $4.0\,\mathrm{m\,s^{-1}}$. The ball then bounces vertically upwards. Figure **2.77** is the graph of velocity against time for the ball. The positive direction for velocity is upwards.

a Find the magnitude of the momentum change of the ball during the bounce.

b The ball stayed in contact with the floor for 0.15 s. What average force did the floor exert on the ball?

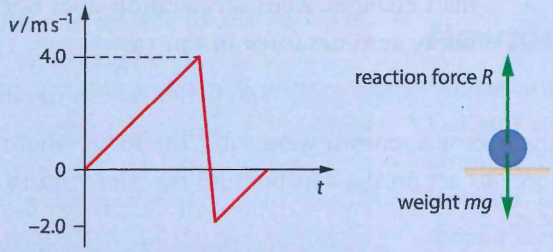


Figure 2.77

a The momentum when the ball hits the floor is: $0.50 \times 4.0 = 2.0\,\mathrm{N\,s}$

The momentum when the ball rebounds from the floor is: $0.50 \times (-2.0) = -1.0\,\mathrm{N\,s}$

The magnitude of the momentum change is therefore 3.0 N s.

b The forces on the ball are its weight and the reaction from the floor, R.

$$F_{net} = R - mg$$

This is also the force that produces the change in momentum:

$$F_{net} = \frac{\Delta p}{\Delta t}$$

Substituting in this equation:

$$F_{net} = \frac{3.0}{0.15} = 20\,\mathrm{N}$$

We need to find R, so:

$R = 20 + 5.0 = 25\,\mathrm{N}$.

The average force exerted on the ball by the floor is 25 N.

Exam tip

This is a very tricky problem with lots of possibilities for error. A lot of people forget to include the minus sign in the rebound velocity and also forget the weight, so they answer incorrectly that $R = 20\,\mathrm{N}$.

Impulse and force–time graphs

We may rearrange the equation:

$$F_{\text{net}} = \frac{\Delta p}{\Delta t}$$

to get:

$$\Delta p = F_{\text{net}} \Delta t$$

The quantity $F_{\text{net}}\Delta t$ is called the **impulse** of the force, and is usually denoted by J. It is the product of the average force times the time for which the force acts. The impulse is also equal to the change in momentum. Notice that impulse is a vector whose direction is the same as that of the force (or the change in momentum).

When you jump from a height of, say, 1 m, you will land on the ground with a speed of about $4.5\,\text{m s}^{-1}$. Assuming your mass is 60 kg, your momentum just before landing will be 270 N s and will become zero after you land. From $F_{\text{net}} = \frac{\Delta p}{\Delta t}$, this can be achieved with a small force acting for a long time or large force acting for a short time. You will experience the large force if you do not bend your knees upon landing – keeping your knees stiff means that you will come to rest in a short time. This means Δt will be very small and the force large (which may damage your knees).

The three graphs of Figure **2.78** show three different force–time graphs. Figure **2.78a** shows a (non-constant) force that increases from zero, reaches a maximum value and then drops to zero again. The force acted for a time interval of about 2 ms. The impulse is the area under the curve. Without calculus we can only estimate this area by tediously counting squares: each small square has area $0.1\,\text{ms} \times 0.2\,\text{N} = 2 \times 10^{-5}\,\text{N s}$. There are about 160 full squares under the curve and so the impulse is $3 \times 10^{-3}\,\text{N s}$. (In this case it is not a bad approximation to consider the shape under the curve to be a triangle but with a base of 1.3 ms so that the area is then $\frac{1}{2} \times 1.3 \times 10^{-3} \times 4 \approx 3 \times 10^{-3}\,\text{N s}$.)

In the second graph, the force is constant (Figure **2.78b**). The impulse of the force is $6.0 \times (8.0 - 2.0) = 36\,\text{N s}$. Suppose this force acts on a body of mass 12 kg, initially at rest. Then the speed v of the body after the force stops acting can be found from:

$$\Delta p = 36\,\text{N s}$$

$$mv - 0 = 36\,\text{N s}$$

$$v = \frac{36}{12} = 3.0\,\text{m s}^{-1}$$

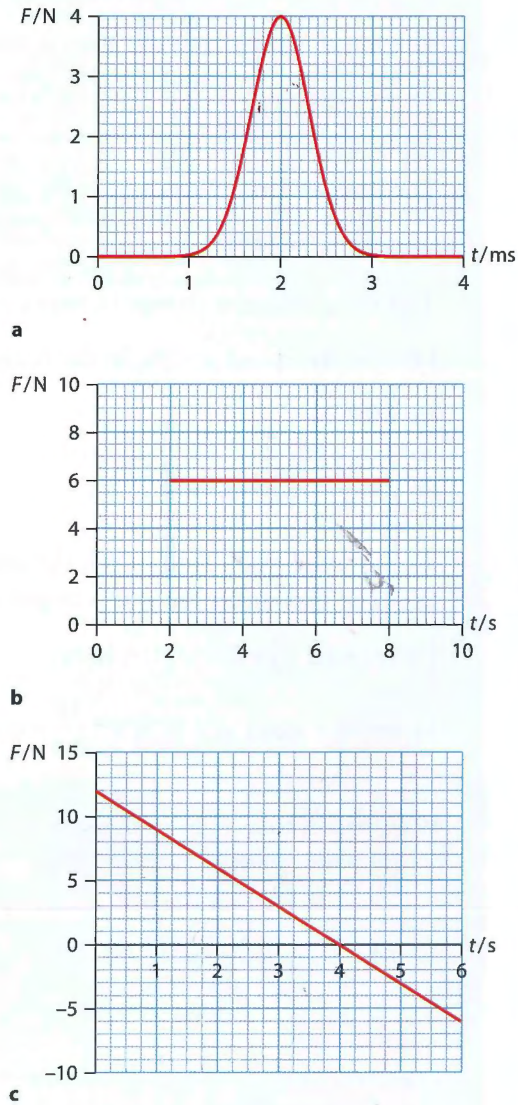


Figure 2.78 Three different force–time graphs: **a** non-constant force, **b** constant force; **c** force that varies linearly with time.

Worked examples

2.48 Consider the graph of Figure **2.78c**. The force acts on a body of mass 3.0 kg initially at rest. Calculate:

a the initial acceleration of the body

b the speed at 4.0 s

c the speed at 6.0 s.

a The initial acceleration a is at $t=0$, when $F=12\,\text{N}$.

$$a=\frac{F}{m}=\frac{12}{3.0}=4.0\,\text{m s}^{-2}$$

b The impulse from 0 s to 4.0 s is the area under this part of the graph:

$$\tfrac{1}{2}\times 4.0\times 12=24\,\text{N s}$$

This is equal to the change in momentum.

Let v be the speed at 4.0 s. As the body is initially at rest, the momentum change is:

$$mv-0=24$$

So $v=\dfrac{24}{m}=\dfrac{24}{3.0}=8.0\,\text{m s}^{-1}$

c The impulse from 0 s to 6.0 s is the area under the graph, which includes part above the axis and part below the axis. The part under the axis is negative, as the force is negative here, so the impulse is:

$$\tfrac{1}{2}\times 4.0\times 12-\tfrac{1}{2}\times 2.0\times 6.0=18\,\text{N s}$$

Hence the speed at 6.0 s is $v=\dfrac{18}{3.0}=6.0\,\text{m s}^{-1}$.

2.49 A ball of mass 0.20 kg moving at $3.6\,\text{m s}^{-1}$ on a horizontal floor collides with a vertical wall. The ball rebounds with a speed of $3.2\,\text{m s}^{-1}$. The ball was in contact with the wall for 12 ms. Determine the maximum force exerted on the ball, assuming that the force depends on time according to Figure **2.79**.

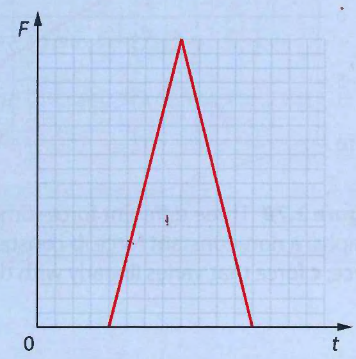


Figure 2.79

Let the initial velocity be positive. The rebound velocity is then negative.

Initial momentum: $0.20 \times 3.6 = 0.72\,\text{N s}$

Final momentum: $0.20 \times (-3.2) = -0.64\,\text{N s}$

The change in momentum of the ball is:

$$-0.64 - 0.72 = -1.36\,\text{N s}$$

The magnitude of the change in momentum is equal to the area under the force–time graph.

The area is $\frac{1}{2} \times 12 \times 10^{-3} \times F_{max}$ and so:

$$\frac{1}{2} \times 12 \times 10^{-3} \times F_{max} = 1.36\,\text{N s}$$

$$\Rightarrow F_{max} = 0.227 \times 10^{3} \approx 2.3 \times 10^{2}\,\text{N}$$

Conservation of momentum

Consider a system with momentum p. The net force on the system is:

$$F_{net} = \frac{\Delta p}{\Delta t}$$

and so if $F_{net} = 0$ it follows that $\Delta p = 0$. There is no change in momentum. This is expressed as the law of **conservation of momentum**:

When the net force on a system is zero the momentum does not change, i.e. it stays the same. We say it is conserved.

Notice that 'system' may refer to a single body or a collection of many different bodies.

Let us consider the blue block of mass 4.0 kg moving at speed $6.0\,\text{m s}^{-1}$ to the right shown in Figure **2.80**. The blue block collides with the red block of mass 8.0 kg that is initially at rest. After the collision the two blocks move off together.

As the blocks collide, each will exert a force on the other. By Newton's third law, the magnitude of the force on each block is the same. There are no forces that come from outside the system, i.e. no external forces. You might say that the weights of the blocks are forces that come from the outside. That is correct, but the weights are cancelled by the normal reaction forces from the table. So the net external force on the system is zero. Hence we expect that the total momentum will stay the same.

The total momentum before the collision is:

$$4.0 \times 6.0 + 8.0 \times 0 = 24\,\text{N s}$$

The total momentum after the collision is:

$$(4.0 + 8.0) \times v = 12v$$

where v is the common speed of the two blocks.

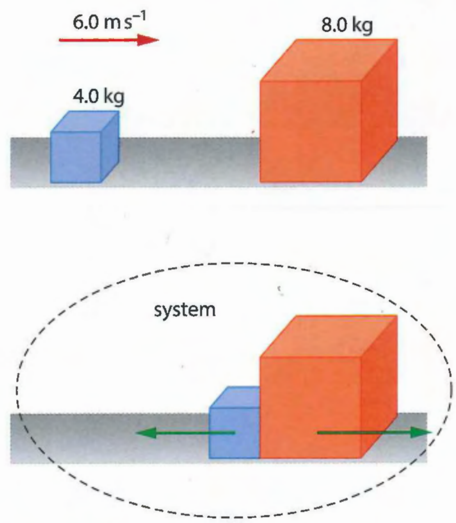


Figure 2.80 In a collision with no external forces acting, the total momentum of the system stays the same.

Equating the momentum after the collision and the momentum before the collision:

$$12v = 24$$

$$\Rightarrow \quad v = 2.0\,\mathrm{m\,s^{-1}}$$

The kinetic energy before the collision is:

$$\tfrac{1}{2} \times 4.0 \times 6.0^2 = 72\,\mathrm{J}$$

After the collision the kinetic energy is:

$$\tfrac{1}{2} \times 12 \times 2.0^2 = 24\,\mathrm{J}$$

It appears that 48 J has been 'lost' (into other forms of energy, e.g. thermal energy in the blocks themselves and the surrounding air or energy to deform the bodies during the collision and some to sound generated in the collision).

But consider now the outcome of the collision of these two blocks in which the blue block rebounds with speed $2.0\,\mathrm{m\,s^{-1}}$, as shown in Figure **2.81**. The red block moves off in the original direction with speed v.

What is the speed of the red block? As before, the total momentum before the collision is 24 N s. The total momentum after the collision is (watch the minus sign):

$$\underbrace{(4.0 \times -2.0)}_{\text{blue block}} + \underbrace{(8.0 \times v)}_{\text{red block}}$$

Equating the total momentum before and after the collision we find:

$$-8.0 + 8.0 \times v = 24$$

This gives $v = 4.0\,\mathrm{m\,s^{-1}}$.

The total kinetic energy after the collision is then:

$$\underbrace{\tfrac{1}{2} \times 4.0 \times (-2.0)^2}_{\text{blue block}} + \underbrace{\tfrac{1}{2} \times 8.0 \times 4.0^2}_{\text{red block}} = 72\,\mathrm{J}$$

This is the same as the initial kinetic energy.

So, in a collision the momentum is always conserved but kinetic energy may or may not be conserved. You will find out more about this in the next section.

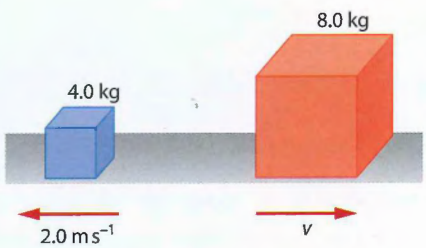


Figure 2.81 An outcome of the collision in which total kinetic energy stays the same.

Predicting outcomes

Physics is supposed to be able to predict outcomes. So why is there more than one outcome in the collision of Figure **2.80**? Physics does predict what happens, but more information about the nature of the colliding bodies is needed. We need to know if they are soft or hard, deformable or not, sticky or breakable, etc. If this information is given physics will uniquely predict what will happen.

Kinetic energy and momentum

We have seen that, in a collision or explosion where no external forces are present, the total momentum of the system is conserved. You can easily convince yourself that in the three collisions illustrated in Figure **2.82** momentum is conserved. The incoming body has mass 8.0 kg and the other a mass of 12 kg.

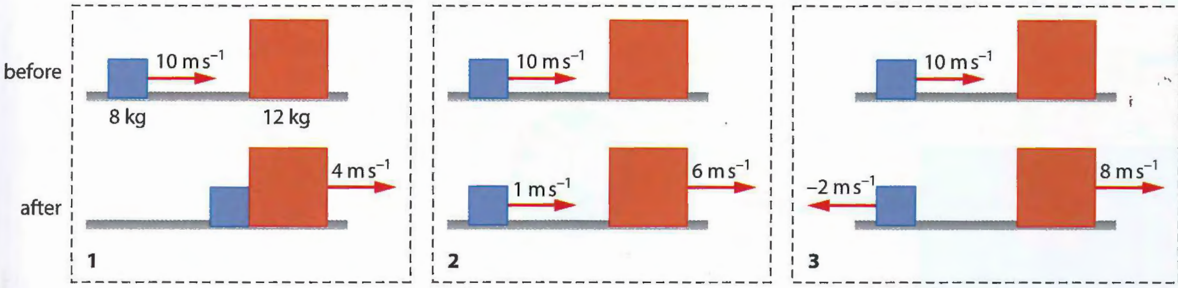


Figure 2.82 Momentum is conserved in these three collisions.

Let us examine these collisions from the point of view of energy. In all cases the total kinetic energy before the collision is:

$$E_K = \frac{1}{2} \times 8.0 \times 10^2 = 400\,\text{J}$$

The total kinetic energy after the collision in each case is:

case 1: $E_K = \frac{1}{2} \times 20 \times 4^2 = 160\,\text{J}$

case 2: $E_K = \frac{1}{2} \times 8.0 \times 1^2 + \frac{1}{2} \times 12 \times 6^2 = 220\,\text{J}$

case 3: $E_K = \frac{1}{2} \times 8.0 \times 2^2 + \frac{1}{2} \times 12 \times 8^2 = 400\,\text{J}$

We thus observe that whereas momentum is conserved in all cases, kinetic energy is not. When kinetic energy is conserved (case 3), the collision is said to be **elastic**. When it is not (cases 1 and 2), the collision is *inelastic*. In an inelastic collision, kinetic energy is lost. When the bodies stick together after a collision (case 1), the collision is said to be **totally inelastic** (or **plastic**), and in this case the maximum possible kinetic energy is lost.

The lost kinetic energy is transformed into other forms of energy, such as thermal energy, deformation energy (if the bodies are permanently deformed as a result of the collision) and sound energy.

Notice that using momentum, we can obtain a useful additional formula for kinetic energy:

$$E_K = \frac{1}{2} m v^2 = \frac{m^2 v^2}{2m}$$

$$E_K = \frac{p^2}{2m}$$

Worked examples

2.50 A moving body of mass m collides with a stationary body of double the mass and sticks to it. Calculate the fraction of the original kinetic energy that is lost.

The original kinetic energy is $\frac{1}{2}mv^2$ where v is the speed of the incoming mass. After the collision the two bodies move as one with speed u that can be found from momentum conservation:

$$mv = (m + 2m)u$$

$$\Rightarrow \quad u = \frac{v}{3}$$

The total kinetic energy after the collision is therefore:

$$\frac{1}{2}(3m) \times \left(\frac{v}{3}\right)^2 = \frac{mv^2}{6}$$

and so the lost kinetic energy is

$$\frac{mv^2}{2} - \frac{mv^2}{6} = \frac{mv^2}{3}$$

The fraction of the original energy that is lost is thus

$$\frac{mv^2/3}{mv^2/2} = \frac{2}{3}$$

2.51 A body at rest of mass M explodes into two pieces of masses $M/4$ and $3M/4$. Calculate the ratio of the kinetic energies of the two fragments.

Here it pays to use the formula for kinetic energy in terms of momentum: $E_K = \frac{p^2}{2m}$. The total momentum before the explosion is zero, so it is zero after as well. Thus, the two fragments must have equal and opposite momenta. Hence:

$$\frac{E_{light}}{E_{heavy}} = \frac{p^2 / (2M_{light})}{(-p)^2 / (2M_{heavy})}$$

$$\frac{E_{light}}{E_{heavy}} = \frac{M_{heavy}}{M_{light}}$$

$$\frac{E_{light}}{E_{heavy}} = \frac{3M/4}{M/4}$$

$$\frac{E_{light}}{E_{heavy}} = 3$$

It all depends on the system!

Consider a ball that you drop from rest from a certain height. As the ball falls, its speed and hence its momentum increases so momentum does not stay the same (Figure **2.83**).

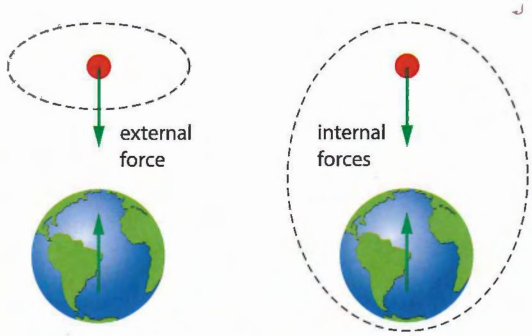


Figure 2.83 As the ball falls, an external force acts on it (its weight), increasing its momentum.

This is to be expected – there is an external force on the ball, namely its weight. So the momentum of the system that consists of just the falling ball is not conserved. If we include the Earth as part of the system then there are no external forces and the total momentum will be conserved. This means that the Earth moves up a bit as the ball falls!

The rocket equation

The best example of motion with varying mass is, of course, the rocket (Figure **2.84**).

Figure 2.84 Exhaust gases from the booster rockets propel this space shuttle during its launch.

This is quite a complex topic and is included here only as supplementary material. The rocket moves with speed v. The engine is turned on and gases leave the rocket with speed u relative to the rocket. The initial mass of the rocket including the fuel is M. After a short time δt the rocket has ejected fuel of mass δm. The mass of the rocket is therefore reduced to $M-\delta m$ and its speed increased to $v+\delta v$ (Figure **2.85**).

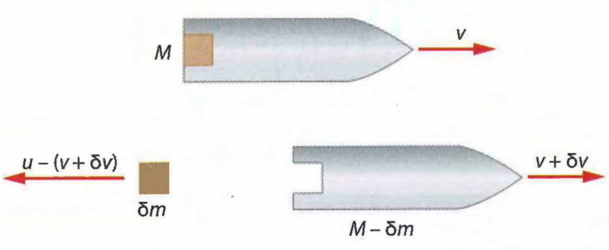


Figure 2.85 Diagram for deriving the rocket equation. The velocities are relative to an observer 'at rest on the ground'.

Applying the law of conservation of momentum gives (in the equation below terms shaded the same colour cancel out):

$$Mv = (M - \delta m)(v + \delta v) - \delta m \underbrace{(u - v - \delta v)}_{\text{speed relative to ground}}$$

$$Mv = Mv + M\delta v - v\delta m - \delta m\delta v - u\delta m + v\delta m + \delta m\delta v$$

$$M\delta v = u\delta m$$

$$\delta v = \frac{\delta m}{Mu}$$

This gives the change in speed of the rocket as a result of gases leaving with speed u relative to the rocket. At time t the mass of the rocket is M. Dividing by δt and taking the limit as δt goes to zero gives the rocket differential equation:

$$M\frac{dv}{dt} = \mu u$$

where μ is the rate at which mass is being ejected.

Nature of science

General principles such as the conservation of momentum allow for simple and quick solutions to problems that may otherwise look complex. Consider, for example, a man of mass m who stands on a plank also of mass m. There is no friction between the floor and the plank. A man starts walking on the plank until he get gets to the other end, at which point he stops. What happens to the plank?

The centre of mass must remain in the same place since there is no external force. So the final position of the plank will be as shown in Figure **2.86**: the plank moves half its length to the left and stops.

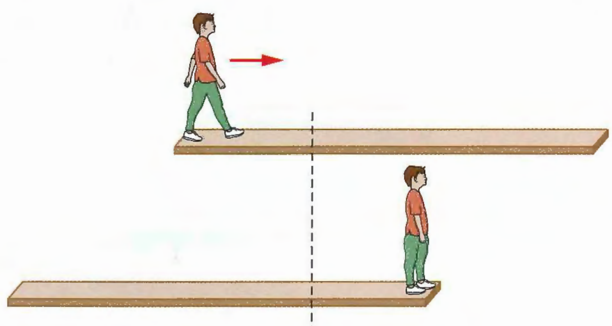


Figure 2.86 Conservation of momentum.

The same principles can be extended to analyse and predict the outcomes of a wide range of physical interactions, from large-scale motion to microscopic collisions.

Test yourself

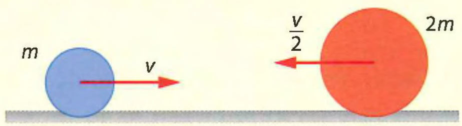
72 The momentum of a ball increased by 12.0 N s as a result of a force that acted on the ball for 2.00 s. Find the average force on the ball.

73 A 0.150 kg ball moving horizontally at 3.00 m s^{-1} collides normally with a vertical wall and bounces back with the same speed.

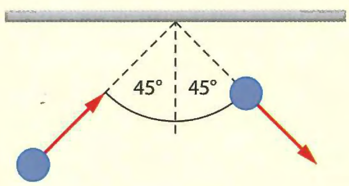
a Calculate the impulse delivered to the ball.

b The ball was in contact with the wall for 0.125 s. Find the average force exerted by the ball on the wall.

74 The bodies in the diagram suffer a head-on collision and stick to each other afterwards. Find their common velocity.



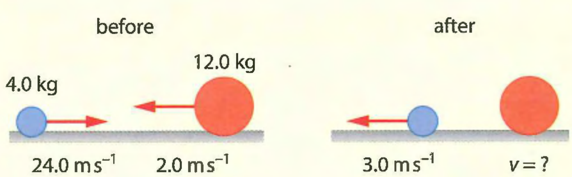
75 A ball of mass 250 g rolling on a horizontal floor with a speed 4.00 m s^{-1} hits a wall and bounces with the same speed, as shown in the diagram.



a What is the magnitude and direction of the momentum change of the ball?

b Is momentum conserved here? Why or why not?

76 Two masses moving in a straight line towards each other collide as shown in the diagram. Find the velocity (magnitude and direction) of the heavier mass after the collision.

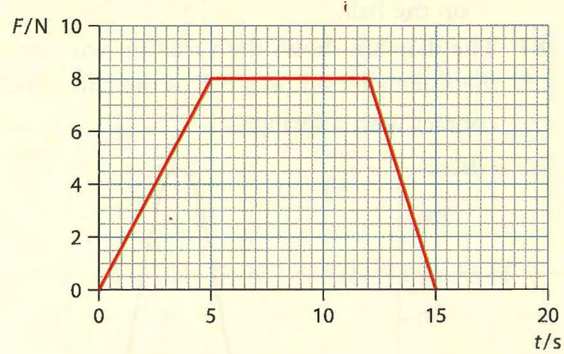


77 A time-varying force varies with time as shown in the graph. The force acts on a body of mass 4.0 kg.

a Find the impulse of the force from $t = 0$ to $t = 15$ s.

b Find the speed of the mass at 15 s, assuming the initial velocity was zero.

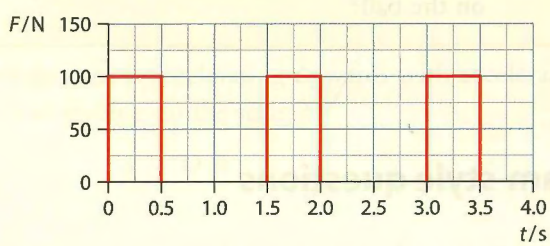
c State the initial velocity of the body such it is brought to rest at 15 s.



78 A boy rides on a scooter pushing on the road with one foot with a horizontal force that depends on time, as shown in the graph. While the scooter rolls, a constant force of 25 N opposes the motion. The combined mass of the boy and scooter is 25 kg.

a Find the speed of the boy after 4.0 s, assuming he started from rest.

b Draw a graph to represent the variation of the boy's speed with time.



79 A ball of mass m is dropped from a height of h_1 and rebounds to a height of h_2. The ball is in contact with the floor for a time interval of t.

a Show that the average net force on the ball is given by:

$$F = m\frac{\sqrt{2gh_1} + \sqrt{2gh_2}}{2}$$

b If $h_1 = 8.0$ m, $h_2 = 6.0$ m, $t = 0.125$ s and $m = 0.250$ kg, calculate the average force exerted by the ball on the floor.

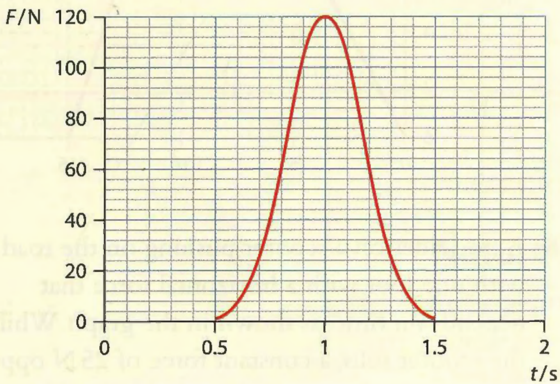
80 A ball of mass m moving vertically, hits a horizontal floor normally with speed v_1 and rebounds with speed v_2. The ball was in contact with the floor for a time t.

a Show that the average force F on the ball from the floor during the collision is given by:

$$F = \frac{m(v_1 + v_2)}{t} + mg$$

b Find an expression for the average net force on the ball.

81 The diagram shows the variation with time of the force exerted on a ball as the ball came into contact with a spring.



a For how long was the spring in contact with the ball?

b Estimate the magnitude of the change in momentum of the ball.

c What was the average force that was exerted on the ball?

82 Two masses of 2.0 kg and 4.0 kg are held in place, compressing a spring between them. When they are released, the 2.0 kg moves away with a speed of $3.0\,\mathrm{m\,s^{-1}}$. What was the energy stored in the spring?

83 A rocket in space where gravity is negligible has a mass (including fuel) of 5000 kg. It is desired to give the rocket an average acceleration of $15.0\,\mathrm{m\,s^{-2}}$ during the first second of firing the engine. The gases leave the rocket at a speed of $1500\,\mathrm{m\,s^{-1}}$ (relative to the rocket). Estimate how much fuel must be burnt in that second.