1 Fundamental Concepts. Vectors

1.1 Introduction

In any scientific theory, and in mechanics in particular, it is necessary to begin with certain primitive concepts. It is also necessary to make a certain number of reasonable assumptions. Two of the most basic concepts are *space* and *time*. In our initial study of the science of motion, mechanics, we shall assume that the physical space of ordinary experience is adequately described by the three-dimensional mathematical space of Euclidean geometry. And with regard to the concept of time, we shall assume that an ordered sequence of events can be measured on a uniform absolute time scale. We shall further assume that space and time are distinct and independent entities. According to the theory of relativity, space and time are not absolute and independent. However, this is a matter to be taken up after the study of the classical foundations of mechanics.

In order to define the position of a body in space, it is necessary to have a reference system. In mechanics we use a *coordinate system*. The basic type of coordinate system for our purpose is the *Cartesian* or *rectangular* coordinate system, a set of three mutually perpendicular straight lines or *axes*. The position of a point in such a coordinate system is specified by three numbers or coordinates, x, y, and z. The coordinates of a *moving* point change with time; that is, they are functions of the quantity t as measured on our time scale.

A very useful concept in mechanics is the *particle* or mass point, an entity that has mass[1] but does not have spatial extension. Strictly speaking the particle is an idealization that may not exist—even the proton has a finite size—but the idea is

[1] The concept of mass will be discussed in Chapter 2.

useful as an approximation of a small body, or rather, one whose size is relatively unimportant in a particular discussion. The earth, for example, might be treated as a particle in celestial mechanics.

1.2 Physical Quantities and Units

The observational data of physics are expressed in terms of certain fundamental entities called *physical quantities*—for example, length, time, force, and so forth. A physical quantity is something that can be measured quantitatively in relation to some chosen unit. When we say that the length of a certain object is, say 7 in., we mean that the quantitative measure 7 is the relation (ratio) of the length of that object to the length of the unit (1 in.). It has been found that it is possible to define all of the *unit* physical quantities of mechanics in terms of just three basic ones, namely *time, length,* and *mass*.

The Unit of Time

The basic unit for measurement of time is the *second*. It is defined in terms of the cesium atomic clock frequency standard, namely, the time required for exactly 9,192,631,770 oscillations of a particular atomic transition of the isotope cesium 133. Prior to the year 1967 the second was defined in terms of the earth's rotational period; that is, the second was equal to 1/86,400 of a mean solar day. This was abandoned in favor of the atomic standard because the earth's period of rotation is not constant.

The Unit of Length

The standard unit of length is the *meter*. This unit is now specified in terms of the velocity of light: The meter is the distance that light travels during a time interval of exactly 1/299,792,458 second. Put in another way, this establishes the velocity of light as precisely 299,792,458 meters/second. Furthermore, since the second is defined in terms of the cesium atomic clock, *both* the meter and the second are atomic-based standards. From 1967 to 1983 the meter was defined in terms of the wavelength of a certain orange spectral line of a krypton 86 lamp. Previous to 1967 the meter was defined to be the distance between two marks on a bar of a platinum-iridium alloy that was stored at the Bureau of Metric Standards, Sevres, France.

The Unit of Mass

The standard unit of mass is the *kilogram*. It is the mass of a cylinder of platinum-iridium also kept at the Bureau of Metric Standards. Copies of this primary standard are owned by the governments of most major countries of the world.

The above units comprise the basis for the Système International d'Unites or SI system.[2] The modern atomic standards of length and time in this system are not only

[2]Other basic and derived units are listed in Appendix A.

more precise than the former standards, but they are also universally reproducible and indestructible. Unfortunately, it is not at present technically feasible to employ an atomic standard of mass.

Actually, there is nothing particularly sacred about the physical quantities time, length, and mass as a basic set to define units. Other sets of physical quantities may be used. The so-called gravitational systems employ time, length, and *force*.

In addition to the SI system, there are other systems in common use, namely, the cgs, or centimeter-gram-second, system, and the fps, or foot-pound-second, system. These latter two systems may be regarded as secondary, because their units are specifically defined fractions of the SI units. See Appendix A.

A physical quantity that is completely specified, in appropriate units, by a single number is called a *scalar*. Familiar examples of scalars are density, volume, and temperature. Mathematically, scalars are treated as ordinary real numbers. They obey all the regular rules of algebraic addition, subtraction, multiplication, division, and so on.

There are certain physical quantities that possess a directional characteristic, such as a displacement from one point in space to another. Such quantities require a direction *and* a magnitude for their complete specification. These quantities are called *vectors* if they combine with each other according to the parallelogram rule of addition as discussed in the next section.[3] Besides displacement in space, other familiar examples of vectors are velocity, acceleration, and force. The vector concept and the development of a whole mathematics of vector quantities have proved indispensible to the development of the science of mechanics. The remainder of this chapter will be devoted largely to a study of the mathematics of vectors.

1.3 Notation. Formal Definitions and Rules of Vector Algebra

Vector quantities are denoted in print by boldface type, for example, **A**, whereas ordinary italic type represents scalar quantities. In written work it is customary to use a distinguishing mark, such as an arrow, $\vec{A}$, to designate a vector.

A given vector **A** is specified by stating its magnitude and its direction relative to some chosen reference system. A vector is represented diagrammatically by a directed line segment, as shown in Figure 1.1. A vector can also be specified by listing its *components* or projections along the coordinate axes. The component symbol $[A_x, A_y, A_z]$ will be used as an alternate designation of a vector. The equation

$$\mathbf{A} = [A_x, A_y, A_z]$$

means that the vector **A** is expressed on the right in terms of its components in a particular coordinate system. (It will be assumed that a Cartesian coordinate system is meant, unless stated otherwise.) For example, if the vector **A** represents a *displacement*

[3] An example of a directed quantity that does not obey the rule for addition is a finite rotation of an object about a given axis. The reader can readily verify that two successive rotations about *different* axes do not produce the same effect as a single rotation determined by the parallelogram rule. For the present we shall not be concerned with such non-vector directed quantities, however.

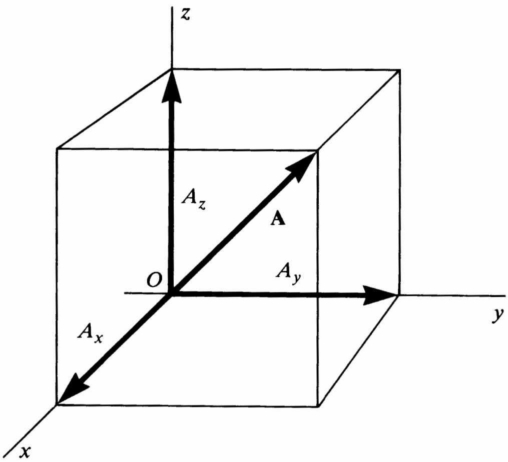

FIGURE 1.1 A vector **A** and its components in Cartesian coordinates.

from a point $P_1(x_1, y_1, z_1)$ to the point $P_2(x_2, y_2, z_2)$, then $A_x = x_2 - x_1$, $A_y = y_2 - y_1$, $A_z = z_2 - z_1$. If **A** represents a *force,* then A_x is the x component of the force, and so on. Clearly, the numerical values of the scalar components of a given vector depend on the choice of the coordinate axes.

If a particular discussion is limited to vectors in a plane, only two components are necessary. On the other hand, one can define a mathematical space of any number of dimensions. Thus the symbol $[A_1, A_2, A_3, \ldots, A_n]$ denotes an n-dimensional vector. In this abstract sense a vector is an ordered set of numbers.

We begin the study of vector algebra with some formal statements concerning vectors.

I. *Equality of Vectors*
The equation

$$\mathbf{A} = \mathbf{B}$$

or

$$[A_x, A_y, A_z] = [B_x, B_y, B_z]$$

is equivalent to the three equations

$$A_x = B_x \qquad A_y = B_y \qquad A_z = B_z$$

That is, two vectors are equal if, and only if, their respective components are equal. Geometrically, equal vectors are parallel and have the same length, but they do not necessarily have the same position. Equal vectors are shown in Figure 1.2, where only two components are drawn for clarity. Notice that the vectors form opposite sides of a parallelogram. (Equal vectors are not necessarily equivalent in all respects. Thus two vectorially equal forces acting at *different* points on an object may produce different mechanical effects.)

II. *Vector Addition*
The addition of two vectors is defined by the equation

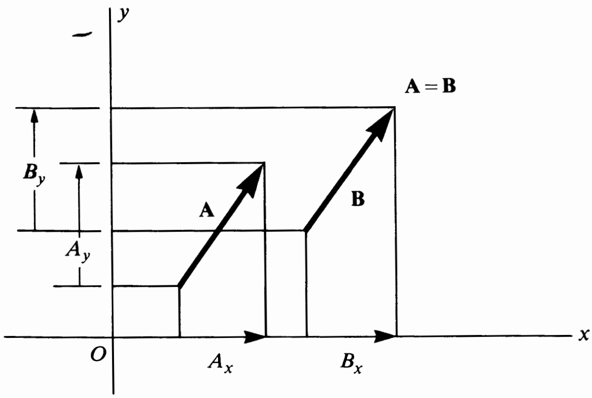

FIGURE 1.2 Illustrating equal vectors.

$$\mathbf{A} + \mathbf{B} = [A_x, A_y, A_z] + [B_x, B_y, B_z] = [A_x + B_x, A_y + B_y, A_z + B_z]$$

The sum of two vectors is a vector whose components are sums of the components of the given vectors. The geometric representation of the vector sum of two non-parallel vectors is the *third* side of a triangle, two sides of which are the given vectors. The vector sum is illustrated in Figure 1.3. The sum is also given by the parallelogram rule, as shown in the figure. The vector sum is defined, however, according to the above equation even if the vectors do not have a common point.

III. *Multiplication by a Scalar*
If c is a scalar and $\mathbf{A}$ a vector,

$$c\mathbf{A} = c[A_x, A_y, A_z] = [cA_x, cA_y, cA_z] = \mathbf{A}c$$

The product $c\mathbf{A}$ is a vector whose components are c times those of $\mathbf{A}$. Geometrically, the vector $c\mathbf{A}$ is parallel to $\mathbf{A}$ and is c times the length of $\mathbf{A}$. When $c = -1$, the vector $-\mathbf{A}$ is one whose direction is the reverse of that of $\mathbf{A}$, as shown in Figure 1.4.

IV. *Vector Subtraction*
Subtraction is defined as follows:

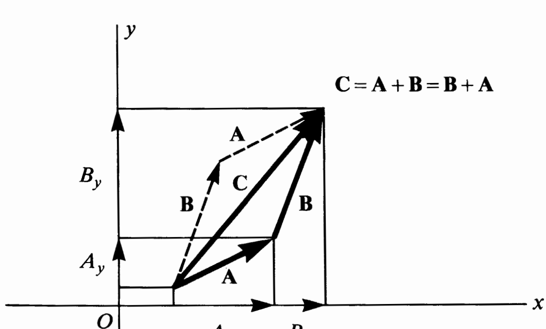

FIGURE 1.3 Addition of two vectors.

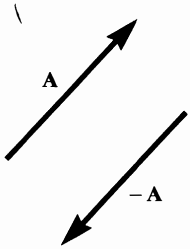

FIGURE 1.4 The negative of a vector.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = [A_x - B_x, A_y - B_y, A_z - B_z]$$

That is, subtraction of a given vector **B** from the vector **A** is equivalent to adding $-\mathbf{B}$ to **A**.

V. *The Null Vector*

The vector $\mathbf{O} = [0, 0, 0]$ is called the *null* vector. The direction of the null vector is undefined. From (IV) it follows that $\mathbf{A} - \mathbf{A} = \mathbf{O}$. Since there can be no confusion when the null vector is denoted by a "zero," we shall hereafter use the notation: $\mathbf{O} = 0$.

VI. *The Commutative Law of Addition*

This law holds for vectors; that is,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

since $A_x + B_x = B_x + A_x$, and similarly for the y and z components.

VII. *The Associative Law*

The associative law is also true, because

$$\begin{aligned}\mathbf{A} + (\mathbf{B} + \mathbf{C}) &= [A_x + (B_x + C_x), A_y + (B_y + C_y), A_z + (B_z + C_z)] \\ &= [(A_x + B_x) + C_x, (A_y + B_y) + C_y, (A_z + B_z) + C_z] \\ &= (\mathbf{A} + \mathbf{B}) + \mathbf{C}\end{aligned}$$

VIII. *The Distributive Law*

Under multiplication by a scalar the distributive law is valid, because, from (II) and (III),

$$\begin{aligned}c(\mathbf{A} + \mathbf{B}) &= c[A_x + B_x, A_y + B_y, A_z + B_z] \\ &= [c(A_x + B_x), c(A_y + B_y), c(A_z + B_z)] \\ &= [cA_x + cB_x, cA_y + cB_y, cA_z + cB_z] \\ &= c\mathbf{A} + c\mathbf{B}\end{aligned}$$

Thus vectors obey the rules of ordinary algebra as far as the above operations are concerned.

IX. *Magnitude of a Vector*

The magnitude of a vector **A**, denoted by $|\mathbf{A}|$ or by A, is defined as the square root of the sum of the squares of the components, namely,

$$A = |\mathbf{A}| = (A_x^2 + A_y^2 + A_z^2)^{1/2}$$

where the positive root is understood. Geometrically, the magnitude of a vector is its length, that is, the length of the diagonal of the rectangular parallelepiped whose sides are A_x, A_y, and A_z, expressed in appropriate units.

X. *Unit Coordinate Vectors*

A *unit vector* is a vector whose magnitude is unity. Unit vectors are often designated by the symbol **e** from the German word *einheit*. The three unit vectors

$$\mathbf{e}_x = [1, 0, 0] \qquad \mathbf{e}_y = [0, 1, 0] \qquad \mathbf{e}_z = [0, 0, 1]$$

are called *unit coordinate vectors* or *basis vectors*. In terms of basis vectors, any vector can be expressed as a vector sum of components as follows:

$$\begin{aligned}\mathbf{A} = [A_x, A_y, A_z] &= [A_x, 0, 0] + [0, A_y, 0] + [0, 0, A_z] \\ &= A_x[1, 0, 0] + A_y[0, 1, 0] + A_z[0, 0, 1] \\ &= \mathbf{e}_x A_x + \mathbf{e}_y A_y + \mathbf{e}_z A_z\end{aligned}$$

A widely used notation for Cartesian unit vectors are the letters **i**, **j**, and **k**, namely

$$\mathbf{i} = \mathbf{e}_x \qquad \mathbf{j} = \mathbf{e}_y \qquad \mathbf{k} = \mathbf{e}_z$$

We shall usually employ this notation hereafter.

The directions of the unit coordinate vectors are defined by the coordinate axes (Figure 1.5). They form a right-handed or a left-handed triad, depending on which type of coordinate system is used. It is customary to use right-handed coordinate systems. The system shown in Figure 1.5 is right-handed.

Examples

1.1 Find the sum and the magnitude of the sum of the two vectors $\mathbf{A} = [1, 0, 2]$ and $\mathbf{B} = [0, 1, 1]$. Adding components we have $\mathbf{A} + \mathbf{B} = [1, 0, 2] + [0, 1, 1] = [1, 1, 3]$

$$|\mathbf{A} + \mathbf{B}| = (1 + 1 + 9)^{1/2} = \sqrt{11}$$ ■

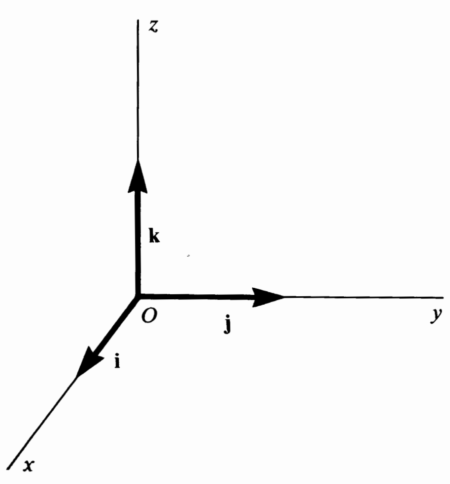

FIGURE 1.5 The unit vectors **ijk**.

1.2 For the above two vectors, express the difference in **ijk** form. Subtracting components, we have

$$\mathbf{A} - \mathbf{B} = [1, -1, 1] = \mathbf{i} - \mathbf{j} + \mathbf{k} \qquad \blacksquare$$

1.3 A helicopter flies 100 m vertically upward, then 500 m horizontally east then 1000 m horizontally north. How far is it from a second helicopter that starts from the same point rising 200 m upward, 100 m west and 500 m north? Solution: Choosing "up", "east", and "north" as basis directions, the final position of the first helicopter is expressed vectorially as $\mathbf{A} = [100, 500, 1000]$ and the second as $\mathbf{B} = [200, -100, 500]$, in meters. Hence the distance between the final positions is given by the expression

$$|\mathbf{A} - \mathbf{B}| = |[(100 - 200), (500 + 100), (1000 - 500)]|\text{m} = (100^2 + 600^2 + 500^2)^{1/2} \text{ m}$$
$$= 787.4 \text{ m} \qquad \blacksquare$$

1.4 The Scalar Product

Given two vectors **A** and **B**, the scalar product or "dot" product, $\mathbf{A} \cdot \mathbf{B}$, is the scalar defined by the equation

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.1}$$

It follows from the above definition that scalar multiplication is *commutative*

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1.2}$$

since $A_x B_x = B_x A_x$, and so on. It also follows that it is *distributive*

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{1.3}$$

because if we apply the definition [(1.1)] in detail

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= A_x(B_x + C_x) + A_y(B_y + C_y) + A_z(B_z + C_z) \\ &= A_x B_x + A_y B_y + A_z B_z + A_x C_x + A_y C_y + A_z C_z \\ &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \end{aligned}$$

From analytical geometry we recall the formula for the cosine of the angle between two line segments

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{(A_x^2 + A_y^2 + A_z^2)^{1/2}(B_x^2 + B_y^2 + B_z^2)^{1/2}} = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} \tag{1.4}$$

or

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \tag{1.5}$$

The above equation may be regarded as an alternate definition for the dot product. Geometrically, $\mathbf{A} \cdot \mathbf{B}$ is equal to the length of the projection of **A** on **B**, times the length of **B**.

If the dot product $\mathbf{A} \cdot \mathbf{B}$ is equal to zero, then **A** is perpendicular to **B**, provided neither **A** nor **B** is null.

The square of the magnitude of a vector **A** is given by the dot product of **A** with itself,

$$A^2 = |\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}$$

From the definitions of the unit coordinate vectors **i**, **j**, and **k**, it is clear that the following relations hold:

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0 \end{aligned} \tag{1.6}$$

Expressing Any Vector as the Product of Its Magnitude by a Unit Vector. Projection.

Consider the equation

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$

Multiply and divide on the right by the magnitude of **A**

$$\mathbf{A} = A\left(\mathbf{i}\frac{A_x}{A} + \mathbf{j}\frac{A_y}{A} + \mathbf{k}\frac{A_z}{A}\right)$$

Now $A_x/A = \cos\alpha$, $A_y/A = \cos\beta$, and $A_z/A = \cos\gamma$ are the *direction cosines* of the vector **A**, and α, β, and γ are the *direction angles*. Thus we can write

$$\mathbf{A} = A(\mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma) = A[\cos\alpha, \cos\beta, \cos\gamma]$$

or

$$\mathbf{A} = A\mathbf{n} \tag{1.7}$$

where **n** is a unit vector whose components are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. Consider any other vector **B**. Clearly, the projection of **B** on **A** is just

$$B\cos\theta = \frac{\mathbf{B} \cdot \mathbf{A}}{A} = \mathbf{B} \cdot \mathbf{n} \tag{1.8}$$

where θ is the angle between **A** and **B**.

Examples

1.4 *Component of a vector. Work*

As an example of the dot product, suppose that an object under the action of a constant force[4] undergoes a linear displacement $\Delta\mathbf{s}$, as shown in Figure 1.6. By definition, the *work* ΔW done by the force is given by the product of the component of the force **F** in the direction of $\Delta\mathbf{s}$, multiplied by the magnitude Δs of the displacement, that is,

$$\Delta W = (F\cos\theta)\,\Delta s$$

[4]The concept of force will be discussed in Chapter 2.

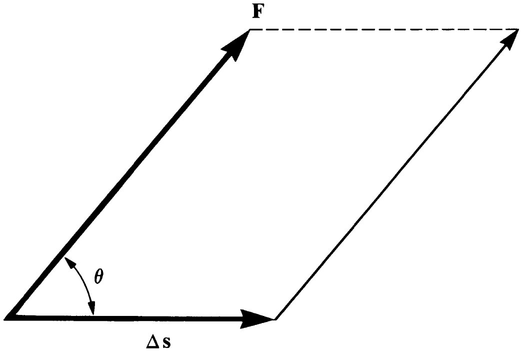

FIGURE 1.6 A force acting on a body undergoing a displacement.

where θ is the angle between **F** and Δ**s**. But the expression on the right is just the dot product of **F** and Δ**s**, that is,

$$\Delta W = \mathbf{F} \cdot \Delta\mathbf{s}$$ ■

1.5 *Law of cosines*

Consider the triangle whose sides are **A**, **B**, and **C**, as shown in Figure 1.7. Then $\mathbf{C} = \mathbf{A} + \mathbf{B}$. Take the dot product of **C** with itself

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$$

$$= \mathbf{A} \cdot \mathbf{A} + 2\mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B}$$

The second step follows from the application of the rules in Equations 1.2 and 1.3. Replace $\mathbf{A} \cdot \mathbf{B}$ by $AB \cos \theta$ to obtain

$$C^2 = A^2 + 2AB \cos \theta + B^2$$

which is the familiar law of cosines. This is just one example of the use of vector algebra to prove theorems in geometry. ■

1.6 Find the cosine of the angle between a long diagonal and an adjacent face diagonal of a cube. Solution: We can represent the two diagonals in question by the vectors $\mathbf{A} = [1, 1, 1]$ and $\mathbf{B} = [1, 1, 0]$. Hence, from Equation 1.4

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{A\,B} = \frac{1 + 1 + 0}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}} = 0.8165$$ ■

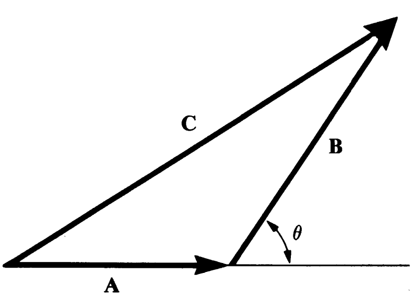

FIGURE 1.7 The law of cosines.

1.7 The vector $a\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. What is the value of a? Solution: If the vectors are perpendicular to each other, their dot product must vanish ($\cos 90° = 0$). Hence, we have

$$(a\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = a + 2 + 3 = a + 5 = 0$$

Hence

$$a = -5$$

■

1.5 The Vector Product

Given two vectors **A** and **B**, the vector product or "cross product," $\mathbf{A} \times \mathbf{B}$, is defined as the vector whose components are given by the equation

$$\mathbf{A} \times \mathbf{B} = [A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x] \tag{1.9}$$

It can be shown that the following rules hold for cross multiplication:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1.10}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1.11}$$

$$n(\mathbf{A} \times \mathbf{B}) = (n\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (n\mathbf{B}) \tag{1.12}$$

The proofs of these follow directly from the definition and are left as an exercise. (*Note:* The first equation states that the cross product is *anticommutative*.)

According to the definitions of the unit coordinate vectors, Section 1.3, it readily follows that

$$\begin{aligned} \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} = -\mathbf{k} \times \mathbf{j} \\ \mathbf{i} \times \mathbf{j} &= \mathbf{k} = -\mathbf{j} \times \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} = -\mathbf{i} \times \mathbf{k} \end{aligned} \tag{1.13}$$

For example,

$$\mathbf{i} \times \mathbf{j} = [0 - 0, 0 - 0, 1 - 0] = [0, 0, 1] = \mathbf{k}$$

The remaining equations are easily proved in a similar manner.

The cross product expressed in **ijk** form is

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) + \mathbf{k}(A_xB_y - A_yB_x)$$

Each term in parentheses is equal to a determinant

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}\begin{vmatrix} A_y A_z \\ B_y B_z \end{vmatrix} + \mathbf{j}\begin{vmatrix} A_z A_x \\ B_z B_x \end{vmatrix} + \mathbf{k}\begin{vmatrix} A_x A_y \\ B_x B_y \end{vmatrix}$$

and finally

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.14}$$

which is readily verified by expansion. The determinant form is a convenient aid for remembering the definition of the cross product. From the properties of determinants, it can be seen at once that if **A** is parallel to **B**, that is, if $\mathbf{A} = c\mathbf{B}$, then the two lower rows of the determinant are proportional and so the determinant is null. Thus the cross product of two parallel vectors is null.

Let us calculate the magnitude of the cross product. We have

$$|\mathbf{A} \times \mathbf{B}|^2 = (A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2$$

With a little patience this can be reduced to

$$|\mathbf{A} \times \mathbf{B}|^2 = (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2$$

or, from the definition of the dot product, the above equation may be written in the form

$$|\mathbf{A} \times \mathbf{B}|^2 = A^2B^2 - (\mathbf{A} \cdot \mathbf{B})^2 \tag{1.15}$$

Taking the square root of both sides of the above equation and using Equation 1.5, we can express the magnitude of the cross product as

$$|\mathbf{A} \times \mathbf{B}| = AB(1 - \cos^2\theta)^{1/2} = AB \sin\theta \tag{1.16}$$

where θ is the angle between **A** and **B**.

To interpret the cross product geometrically, we observe that the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and to **B**, because

$$\begin{aligned} \mathbf{A} \cdot \mathbf{C} &= A_xC_x + A_yC_y + A_zC_z \\ &= A_x(A_yB_z - A_zB_y) + A_y(A_zB_x - A_xB_z) + A_z(A_xB_y - A_yB_x) \\ &= 0 \end{aligned}$$

Similarly, $\mathbf{B} \cdot \mathbf{C} = 0$. Thus the vector **C** is perpendicular to the plane containing the vectors **A** and **B**.

The sense of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is determined from the requirement that the

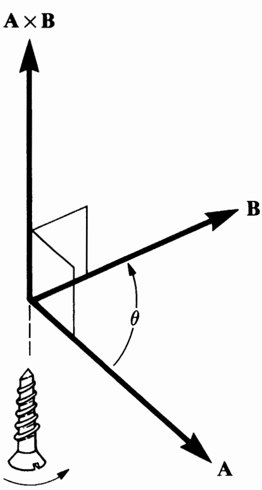

FIGURE 1.8 The cross product of two vectors.

three vectors **A**, **B**, and **C** form a right-handed triad, as shown in Figure 1.8. (This is consistent with the previously established result that in the right-handed triad **ijk** we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.) Therefore, from Equation 1.16 we see that we can write

$$\mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{n} \tag{1.17}$$

where **n** is a unit vector normal to the plane of the two vectors **A** and **B**. The sense of **n** is given by the *right-hand rule,* that is, the direction of advancement of a right-handed screw rotated from the positive direction of **A** to that of **B** through the smallest angle between them, as illustrated in Figure 1.8. Equation 1.17 may be regarded as an alternate definition of the cross product.

Examples

1.8 Given the two vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find $\mathbf{A} \times \mathbf{B}$. In this case it is convenient to use the determinant form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i}(2 - 1) + \mathbf{j}(-1 - 4) + \mathbf{k}(-2 - 1)$$

$$= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k} \quad \blacksquare$$

1.9 Find a unit vector normal to the plane containing the two vectors **A** and **B** above. Solution:

$$\mathbf{n} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}}{[1^2 + 5^2 + 3^2]^{1/2}}$$

$$= \frac{\mathbf{i}}{\sqrt{35}} - \frac{5\mathbf{j}}{\sqrt{35}} - \frac{3\mathbf{k}}{\sqrt{35}} \quad \blacksquare$$

1.6 An Example of the Cross Product: Moment of a Force

A particularly useful application of the cross product is the representation of moments. Let a force **F** act at a point $P(x, y, z)$, as shown in Figure 1.9, and let the vector $\overrightarrow{OP}$ be designated by **r**, that is,

$$\overrightarrow{OP} = \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

The moment **N**, or the *torque vector*, about a given point O is defined as the cross product

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} \tag{1.18}$$

Thus the moment of a force about a point is a vector quantity having a magnitude and a direction. If a single force is applied at a point P on a body that is initially at rest and is free to turn about a fixed point O as a pivot, then the body tends to rotate. The axis of this rotation is perpendicular to the force **F**, and it is also perpendicular to the line OP.